Design of experiments with classical and quantum orthogonal arrays

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Example

A classical example.

Take 4 aces, 4 kings, 4 queens and 4 jacks. Arrange them into 4×4 *array such that:*

- *in every row and column there is only a* ***single*** *card of each suit*
- *in every row and column there is only a* ***single*** *card of each rank*

Example

Two ***mutually orthogonal*** *Latin squares of sizes* $N = 4$ *formes a* ***Graeco-Latin square****:*

A♠	K♥	Q♦	J♣
Q♣	J♦	A♥	K♠
J♥	Q♠	K♣	A♦
K♦	A♣	J♠	Q♥

Example

Euler's problem: 36 officers of six different ranks from six different units come for a military parade. Arrange them in a square such that in each row/each column all uniforms are different.

Example

There exists no mutually orthogonal Latin square for $N = 2, 6$.

Conjectured: Euler; proved: Gaston Terry, 1901.

Definition (Greaco-Latin square/orthogonal Latin square)

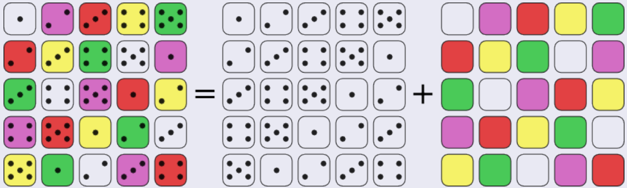
A Greaco-Latin square or orthogonal Latin square of order n over two sets S, T (each consisting n symbols) is $n \times n$ arrangment of cells, each cell containing an ordered pair (s, t) where $s \in S, t \in T$ such that every row and every column contains each element of S and T and that no two cells contain the same ordered pair.

Example

Aα	Bγ	Cδ	Dβ
Bβ	Aδ	Dγ	Cα
Cγ	Dα	Aβ	Bδ
Dδ	Cβ	Bα	Aγ

fjords	jawbox	phlegm	qiviut	zincky
zincky	fjords	jawbox	phlegm	qiviut
qiviut	zincky	fjords	jawbox	phlegm
phlegm	qiviut	zincky	fjords	jawbox
jawbox	phlegm	qiviut	zincky	fjords

Example



Example

0	0
1	1

0	0	0
1	1	0
0	1	1
1	0	1

are orthogonal arrays OA(2,2,2,1) and OA(4,3,2,2).

Example

0	0	0	0
0	1	1	1
0	2	2	2
1	0	1	2
1	1	2	0
1	2	0	1
2	0	2	1
2	1	0	2
2	2	1	0

is orthogonal array OA(9,4,3,2)

Definition (Orthogonal array)

An array A of size $r \times N$ with entries taken from a d-element set S is called orthogonal array $OA(r, N, d, k)$ with r runs, N factors, d levels, k strength and index λ if every $r \times k$ subarray of A contains each k-tuple of symbols from S exactly λ times as a row.

We can obtain a Greaco-Latin square from any orthogonal array: Let's take the last example and set 3^{rd} and 4^{th} columns as the indexing one. In that way we obtain two Latin squares:

0	1	2
2	0	1
1	2	0

0	2	1
2	1	0
1	0	2

Which gives us a Graeco-Latin square:

00	12	12
22	01	10
11	20	02

An $OA(r, N, d, 2)$ is equivalent to $N-2$ Greaco-Latin squares.

Taguchi methods:

- statistical methods (robust design methods) developed by Genichi Taguchi
- main goal: improve the quality, productivity and cost aspects of the process
- later applied to engineering, biotechnology, marketing, advertising

Loss function:

- a function that maps an event/values/... onto a real number representing some "cost" assosiated with them
- an optimization problem seeks to minimize this function
- Taguchi knew statistical theory mainly from Fisher who avoided loss function
- but he realised that there is a need to produce an outcome on target

Taguchi specified 3 situations:

- Larger the better (i.e. agricultural yield)
- Smaller the better (i.e. carbon dioxide emissions)
- On-target, minimum-variation (i.e. mating part in an assembly)

The first two cases are represented by simple monotonic loss function; the third case by a squared-error loss function.

Signal-to-noise (SN) ratio provide a measure of robutstness vs the control factors

SN ratio	goal of the experiment	SN formula
larger the better	maximize the response	$SN = -10\log(\Sigma(1/Y^2)/n)$
smaller the better	minimize the response	$SN = -10\log(\sigma^2)$
on-target	target the response	$SN = -10\log(\Sigma(Y^2)/n))$

The best opportunity to eliminate variation of the final product quality is during the design of a product.
Taguchi developed an off-line quality control which has 3 stages:

- system design: design at the conceptual level
- parameter design (robustification): setting design parameters and nominal values and dimensions to them
- tolerance design: reducing and controlling variation in the critical few dimensions

Innovation of design of experiments:

- designing any task that aims to describe or explain the variation of information under given conditions
- each experiment should be extended with an outer array (orthogonal array): innovationing SN ratio

Eight steps in Taguchi methodology:
Step 1. Identify the main function, side effect and failure mode.
Step 2. Identify the noise factors, testing conditions and quality characteristics.
Step 3. Identify the objective functions to be optimized.
Step 4. Identify the control factors and their levels.
Step 5. Select the orthogonal array matrix experiment.
Step 6. Conduct the matrix experiment.
Step 7. Analyze the data, predict the optimum levels and performance.
Step 8. Perform the verification experiment and plan the future actions.

Example

Step 1. Main function: facing operation on MS work piece using lathe Machine
Side effects: Variation in surface finish

Control factors	*Noise factors*
Cutting speed	*Vibration*
Depth of cut	*Raw material variation*
Feed rate	*Machine condition*
Noise radius	*Temperature*
Coolant	*Operator Skill*

Example

Step 2.
Quality characteristics: surface finish
Work piece materia: mild steel
Cutting tool: Tungsten, carbide tipped tool
Operating machine: Lathe machine
Testing equipment: portable surface tester

Example

Step 3. Objective function: Smaller the better

SN ratio: $SN = -10\log(\sigma^2)$

Step 4. Factor/Levels table:

Factor \ Levels	***1***	***2***	***3***
Cutting speed (v,rpm)	*960*	*640*	*1280*
Depth of cut (t, mm)	*0.3*	*0.2*	*0.4*
Feed rate (f, mm/min)	*145*	*130*	*160*

Example

Step 5. Degrees of freedom: 1 for mean value and 2×4 *two each for the remaining factors*
Orthogonal array:

No. of experiment \ Factors	***1***	***2***	***3***
1	*1*	*1*	*1*
2	*1*	*2*	*2*
3	*1*	*3*	*3*
4	*2*	*1*	*3*
5	*2*	*2*	*1*
6	*2*	*3*	*2*
7	*3*	*1*	*2*
8	*3*	*2*	*3*
9	*3*	*3*	*1*

Example

Step 6.

Experiment No.	Surface Roughness (R_a, □ m)					
	1	2	3	4	5	Mean
1	2.35	2.43	1.94	2.91	2.77	2.48
2	2.5	3.6	2.66	2.98	2.64	2.876
3	2.43	2.82	4.01	2.96	4.1	3.264
4	2.24	3.38	2.45	4.05	4.79	3.382
5	2.54	3.67	2.70	4.25	4.37	3.506
6	4.76	4.25	3.19	3.36	4.35	3.982
7	2.04	2.49	3.84	1.71	3.79	2.834
8	4.4	2.5	3.15	3.24	3.1	3.278
9	3.94	2.19	2.31	2.44	3.30	3.306

Example

Step 7.

Experiment No.	S/N Ratio (dB)
1	-7.9702
2	-9.2568
3	-10.4539
4	-10.9196
5	-11.0971
6	-12.1010
7	-9.2385
8	-10.4642
9	-9.2941

Example

Full factorial design:

Experiment No.	Parameters			Mean Surface Roughness, R_a(□ m)
	Speed(rpm)	Depth of cut(mm)	Feed (mm/min)	
1	960	0.3	145	2.48
2	960	0.2	145	3.26
3	960	0.4	145	2.57
4	960	0.3	130	2.62
5	960	0.2	130	2.876
6	960	0.4	130	2.87
7	960	0.3	160	2.74
8	960	0.2	160	4.35
9	960	0.4	160	3.264
10	640	0.3	145	3.82
11	640	0.2	145	3.506

Example

12	640	0.4	145	3.41
13	640	0.3	130	2.96
14	640	0.2	130	3.45
15	640	0.4	130	3.982
16	640	0.3	160	3.382
17	640	0.2	160	5.04
18	640	0.4	160	4.25
19	1280	0.3	145	4.02
20	1280	0.2	145	4.03
21	1280	0.4	145	3.306
22	1280	0.3	130	2.834
23	1280	0.2	130	4.14
24	1280	0.4	130	3.3
25	1280	0.3	160	2.6
26	1280	0.2	160	3.278
27	1280	0.4	160	2.76

Example

Parameter	Optimum Value
Speed (rpm)	960
Feed Rate (mm/min)	145
Depth of cut (mm)	0.3

Example

Example of quantum orthogonal Latin square (Vicary, Musto; 2016)

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$
$\|\Psi_-\rangle$	$\|\Xi_-\rangle$	$\|\Xi_+\rangle$	$\|\Psi_+\rangle$
$\|\Psi_+\rangle$	$\|\Xi_+\rangle$	$\|\Xi_-\rangle$	$\|\Psi_-\rangle$

where $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ *denote Bell states and* $|\Xi_+\rangle = \frac{1}{\sqrt{5}}(i\,|0\rangle + 2\,|3\rangle)$, $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2\,|0\rangle + i\,|3\rangle)$ *are other entangled states.*

Example

Four states in each row and column form an ***orthogonal basis*** *in* $\mathcal{H}_4$*!*
Classical combinatorics: discrete set of symbols: $1, 2, \ldots, N$.
Quantum combinatorics: continuous family of states $|\psi\rangle \in \mathcal{H}_N$

Definition

A quantum Latin square of ordernis an n-by-n array of elements of the Hilbert space $\mathbb{C}^n$, such that every row and every column is an orthonormal basis.

Example

0	1
1	0

leads to the Bell state: $\left|\Psi_2^+\right\rangle = |01\rangle + |10\rangle$

0	0	0
1	1	0
0	1	1
1	0	1

leads to a 1-uniform state $|\Phi_3\rangle = |000\rangle + |110\rangle + |011\rangle + |101\rangle$

	orthogonal arrays	multipartite quantum state $\|\Phi\rangle$
r	runs	number of terms in the state
N	factors	numer of qudits
d	levels	dimension d of the subsystem
k	strength	class of entanglement

The table shows provided link between orthogonal array $OA(r, N, d, k)$ and k-uniform states provided OA satisfies additional constraints.

From OA(9,4,3,2) we can get a 2-uniform state of 4 qutrits:

$$\left|\Psi_3^4\right\rangle = |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle + |1202\rangle +$$

$$+ |2022\rangle + |2101\rangle + |2210\rangle$$

We can also encode a Greaco-Latin square (pair of Latin sqaures) obtained from OA(9,4,3,2)

00	12	21
22	01	10
11	20	02

to corresponding quantum code:

$$|\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle$$

$$|\tilde{1}\rangle := |022\rangle + |101\rangle + |210\rangle$$

$$|\tilde{2}\rangle := |011\rangle + |120\rangle + |202\rangle$$

Usages:

- quantum teleportation: 2-qubit Bell state allows one to teleport 1 qubit from A to B
- 3-qubit GHZ state allows to teleport 1 qubit between any users
- quantum codes, relations between AME states and multiunitary matrices, perfects tensors, holographic codes...

Definition (Quantum orthogonal array)

A quantum orthogonal array $QOA(r, N, d, k)$ is an arrangement consisting of r rows composed by N-partite normalized pure quantum states $|\psi_j\rangle \in \mathcal{H}_d^{\otimes N}$ having d internal levels each, such that

$$k \sum_{j=0}^{r-1} Tr_{i_1, i_2, \ldots, i_{N-k}}(|\psi_j\rangle \cdot \langle\psi_j|) = kI_k$$

for every subset of $N - k$ parties $i_1, i_2, \ldots, i_{N-k}$.

In other words: a QOA is an arrangement having N columns, possibly entangled, such that every reduction to k columns defines a POVM (set of positive semidefinite operators such that they sum up to identity).

Example

QOA(4,5,2,2) constructed from classical OA(4,3,2,2) and the quantum Bell basis:

$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert \Phi^+\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert \Psi^+\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert \Psi^-\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert \Phi^-\rangle$

where $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ *and* $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ *denote Bell states.*

OA(4,5,2,2):

0	0	0	0	1
1	1	0	1	0
0	1	1	0	0
1	0	1	1	1

QOA(4,5,2,2):

$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|\Phi^+\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|\Psi^+\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|\Psi^-\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|\Phi^-\rangle$

	orthogonal arrays	quantum orthogonal arrays
r	runs	rows/runs
N	factors	space:$\mathcal{H}_d^{\otimes N}$/factors
d	levels	dimension d of the subsystem/internal levels
k	strength	class of entanglement

The table shows provided link between orthogonal array $OA(r, N, d, k)$ and k-uniform states provided OA satisfies additional constraints.

- we can generalize combinatorical structures and their properties
- possible future work: further properties of QOA
- possible future work: usage of QOA in DOE
- possible future work: other quantum combinatorical structures