Krakow Quantum Informatics Seminar (KQIS)
When the asymmetric cryptography will be outdated?

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Cryptology

Science concerned with data communication and storage in secure and usually secret form.

- Cryptography — secret writing (or other methods of hiding information)
- Cryptanalysis — reading secret messages (without knowledge of encryption key).

Past → Today → Future

- **Classic cryptography** — manipulation (substituting and permuting) of symbols (letters), codebooks.
- **Modern cryptography** — computers and mathematics.
- **Quantum cryptography** — quantum physics and quantum computers.
  - quantum cryptanalysis (Shor's algorithm),
  - key exchange methods,
  - detection of security breeches.
- **Postquantum cryptography** — algorithms resistant to quantum cryptanalysis

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- Public key is used to encrypt and verify signature.
- Private key is used to decrypt and to create digital signature.

History of public-key cryptography

1. Diffie-Hellman key exchange protocol: 1976
2. RSA algorithm: 1977
3. ElGamal cryptosystem: 1985
4. British Intelligence (GCHQ): 1970 – 1973 (classified for 25 years)

Key exchange Diffie-Hellman-Merkle protocol

- First asymmetric cryptography algorithm (1976).
- Invented by Martin Hellman and Whitfield Diffi, and independently by Ralph Merkle.
- Only key exchange.
- Security based on the difficulty of calculating discrete logarithms (Discrete Logarithm Problem).

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Eve (eavesdropper) knows x, y, g, $n \rightarrow$ In order to know a and b she must calculate discrete logarithm.

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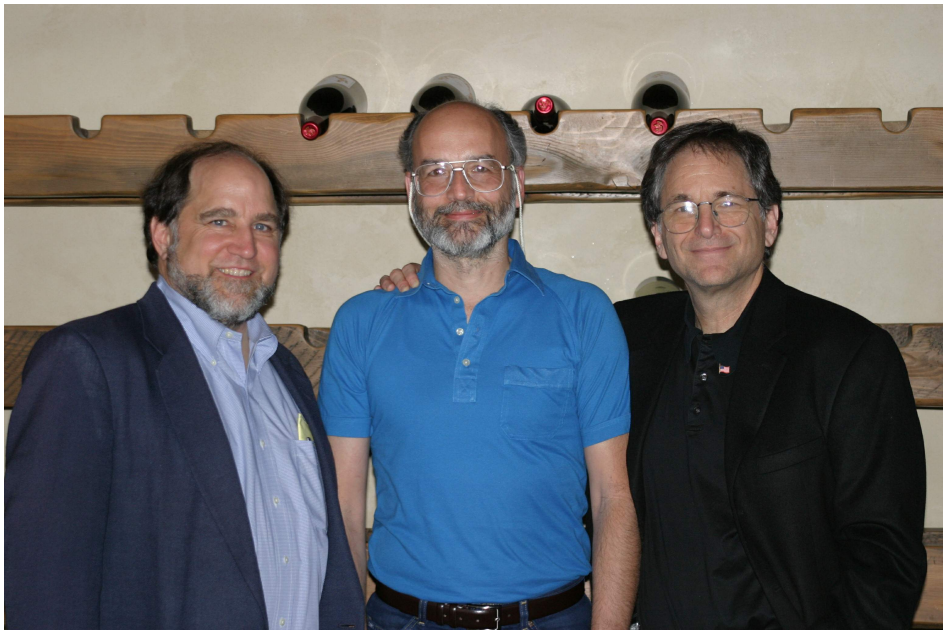

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Private key: (n, d).

Encryption with RSA

1. Message in divided into blocks of size not greater that key size (i.e. 2048 bits). Message is treated as integer number $m_i < n$.
2. Encryption:
   $c_i = m_i^e \mod n$
3. Decryption:
   $m_i = c_i^d \mod n$.
4. RSA encryption in this form is insecure — textbook RSA

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3. Compare $h_m^{ed} = h_m$ and verify the signature.

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5. Quantum computing and Shor algorithm.

Why the RSA is so important?

- Symmetric cryptography is secure but how to exchange keys?
- Public-key cryptography is extremely slow.
- Solution:
  1. public-key cryptography for key exchange/negotiation
  2. symmetric cryptography for data encryption/decryption.
- How to ensure that public key is correct?
  1. Web of Trust like in PGP
  2. Public Key Infrastructure

SSL/TLS protocol

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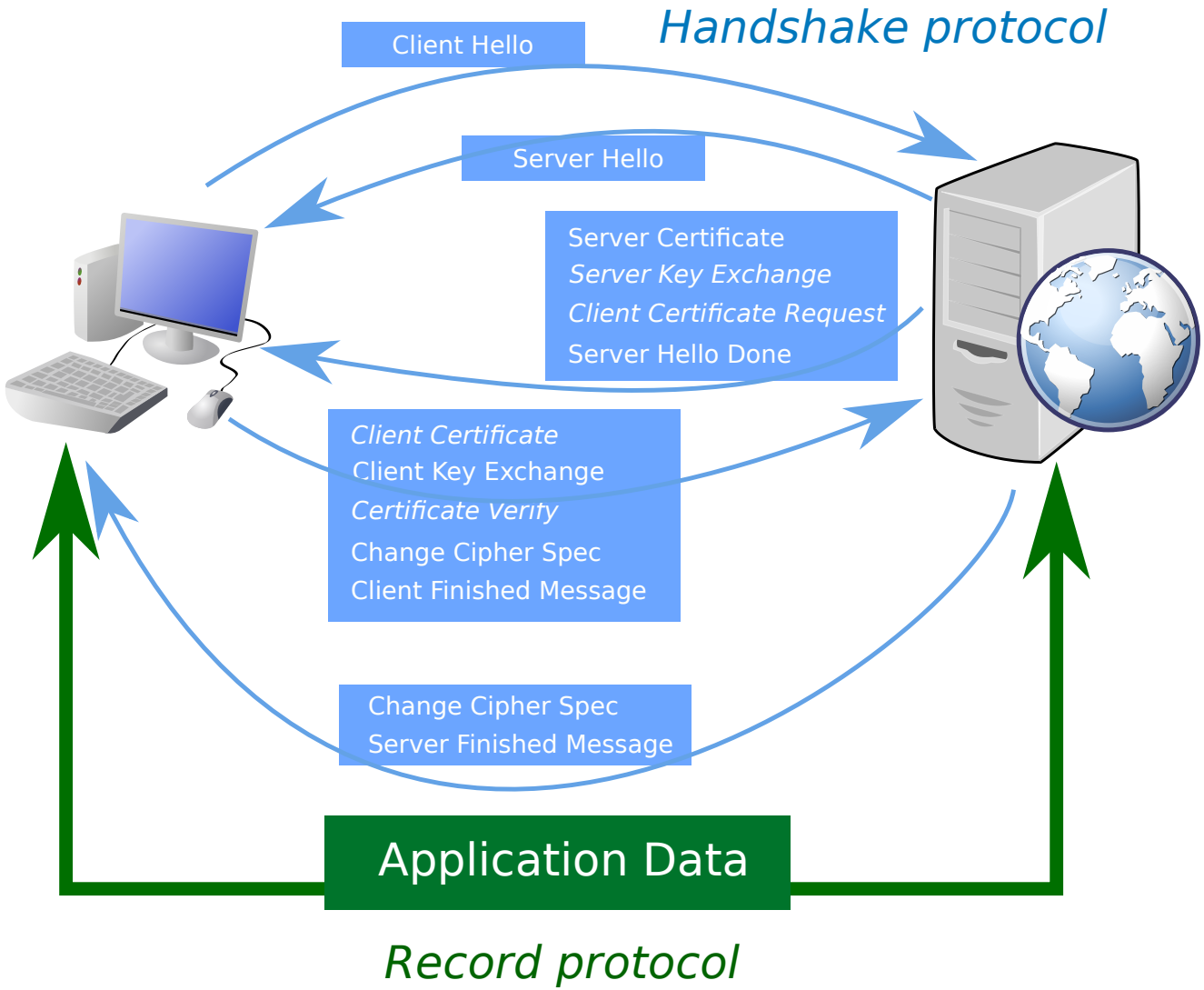
- TLS 1.3, 2018.
- OpenSSL, GnuTLS — open implementations

Handshake protocol w SSL/TLS



Client/Server Hello messages

Client Hello

- Highest supported TLS version.
- 32B string: 4B[client's time] + 28B[random number] — will be used to generate session key.
- Session ID — only if session is restarted.
- Supported cryptographic suite, np. TLS_ECDHE_RSA_WITH _AES_128_GCM_SHA256.

Server Hello

- $\Leftarrow$.
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- Session ID:
  - new ID or
  - ID existing session or
  - null — not supported
- Cryptographic suite. Server chooses the strongest.

Cipher suites

TLS_ECDHE_RSA_WITH_AES_128_GCM_SHA256.

ECDHE — *Elliptic curve Diffie–Hellman* for session key

RSA — for certificates and digital signature.

AES_128_GCM — AES encryption, 128b block, GCM (Galois/Counter Mode) cipher mode.

SHA256 — digest function.

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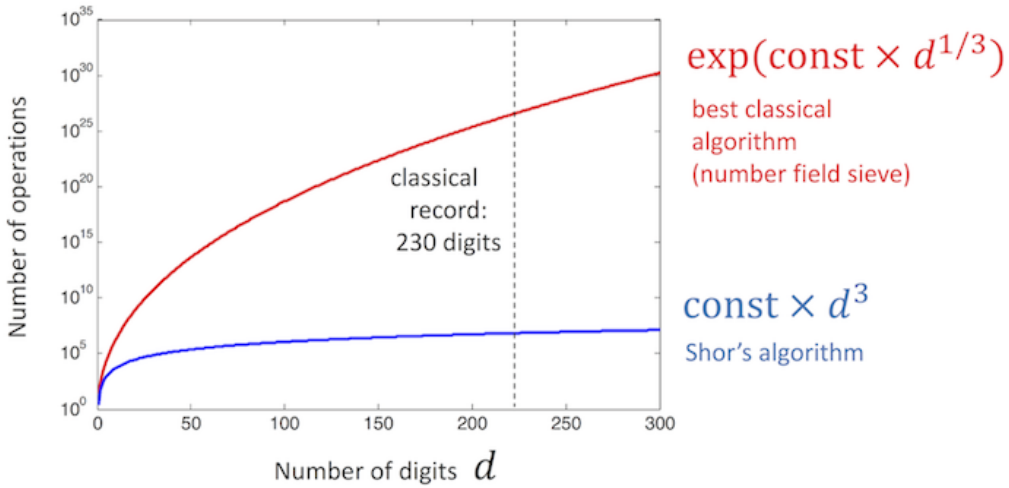
- Shor's algorithm has polynomial complexity



Source: IBM Quantum Experience

Shor algorithm

Peter Shor, 1994

1. Classic (conventional) part: integer factorization to order-finding problem
2. Quantum part: solving order-finding problem

Problem of order-finding

- Modulo operation:

$$a \equiv b \pmod{N}$$

  two integers a, b are congruent modulo N if there is an integer k such that $a - b = kn$
- Let's $\mathbb{Z}_N = \{0, \ldots N-1\}$ be a set defined by operations modulo N
- Let's $\mathbb{Z}_N^* = \{a \in \mathbb{Z}_N : GCD(a, N) = 1\}$ be the multiplicative group of integers modulo N (appropriate operations are defined)
- For $a \in \mathbb{Z}_N^*$ the *order of* a in $\mathbb{Z}_N^*$ is the smallest positive integer r such that:

$$a^r \equiv 1 \pmod{N}$$

Problem of order-finding: example

Let's $N = 15$ and $a = 7$: $7^x \bmod N$

- $7^1 \pmod{15} = 7$
- $7^2 \pmod{15} = 4$
- $7^3 \pmod{15} = 13$
- $7^4 \pmod{15} = 1$ (Eureka!)
- $7^5 \pmod{15} = 16807 \pmod{15} = 7$
- $7^6 \pmod{15} = 117649 \pmod{15} = 4$
- $7^7 \pmod{15} = 823543 \pmod{15} = 13$
- $7^8 \pmod{15} = 5764801 (\mod 15) = 1$ Once again!

then order of 7 in $\mathbb{Z}_{15}^*$ is $r = 4$:
Let's denote the above expression as function $f^7(x) = 7^x \ (mod\ 15)$. It is periodical: $f^7(x + 4) = f^7(x)$.

From integer factorization to order-finding problem

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5. If r is odd go back to step 2.

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5. $a^{\frac{r}{2}} \pm 1$ are not multiply of N but they product is.
6. Prime factors: $p_1(p_2) = GCD(N, a^{\frac{r}{2}} \pm 1)$

Shor algorithm

1. Fourier Transform: retrieve from periodic signal all frequencies
2. Quantum Fourier Transform (QFT) does the same

Shor's algorithm in practice

Table 5: *Quantum factorization records*

Number	# of factors	# of qubits needed	Algorithm	Year implemented	Implemented without prior knowledge of solution
15	2	8	Shor	2001 [2]	×
	2	8	Shor	2007 [3]	×
	2	8	Shor	2007 [3]	×
	2	8	Shor	2009 [5]	×
	2	8	Shor	2012 [6]	×
21	2	10	Shor	2012 [7]	×
143	2	4	minimization	2012 [1]	✓
56153	2	4	minimization	2012 [1]	✓
291311	2	6	minimization	not yet	✓
175	3	3	minimization	not yet	✓

What about other public-key cryptosystems?

- Diffie-Hellman protocol, ElGamal and DSA: *Peter W. Shor, Polynomial-Time Algorithms for Prime Factorization and* ***Discrete Logarithms*** *on a Quantum Computer*
- Elliptic-Curve Cryptography (ECC), ECDH (Elliptic Curve Diffie-Hellman), ECDSA (Elliptic Curve Digital Signature Algorithm): *Quantum Resource Estimates for Computing* ***Elliptic Curve Discrete Logarithms****, Shor's discrete logarithm quantum algorithm for* ***elliptic curves***

Future of cryptography

- Crypto-agility
- Post-quantum cryptography:
  - lattice-based cryptography
  - hash-based cryptography
  - correction codes-based cryptography
  - multivariate cryptography