Software-aided analysis of quantum games

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Game theory

Game theory is the study of mathematical models of strategic interactions among rational agents.

Pure and mixed strategies

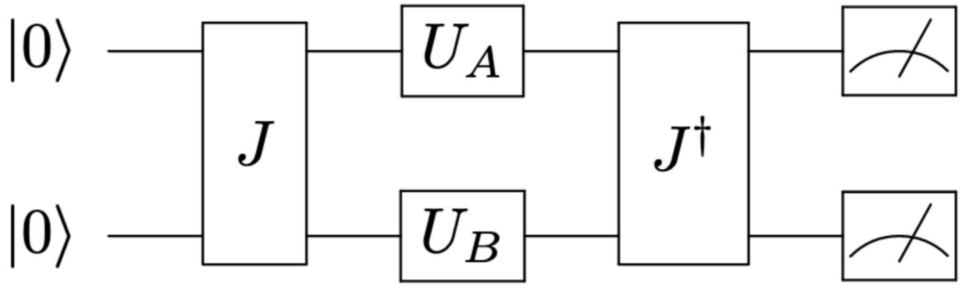
A mixed strategy is an assignment of a probability to each pure strategy.

Prisoner’s Dilemma

The prisoner's dilemma is a standard example of a game analyzed in game theory that shows why two completely rational individuals might not cooperate, even if it appears that it is in their best interests to do so.

	Player 2 cooperates	Player 2 defects
Player 1 cooperates	(-1,-1)	(-3,0)
Player 1 defects	(0,-3)	(-2,-2)

Quantum games – EWL schema



Parameterizations

We can parametrize an arbitrary quantum strategy by using real numbers:

$$U(\theta,\phi) = \begin{bmatrix} e^{i\phi}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & e^{-i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix},\ \theta \in [0,\pi],\ \phi \in [0,\tfrac{\pi}{2}]$$

$$U(\theta,\phi,\alpha) = \begin{bmatrix} e^{-i\phi}\cos\left(\frac{\theta}{2}\right) & e^{i\alpha}\sin\left(\frac{\theta}{2}\right) \\ -e^{-i\alpha}\sin\left(\frac{\theta}{2}\right) & e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix},\ \theta \in [0,\pi],\ \phi,\alpha \in [-\pi,\pi]$$

$$U(\theta,\alpha,\beta) = \begin{bmatrix} e^{i\alpha}\cos\left(\frac{\theta}{2}\right) & ie^{i\beta}\sin\left(\frac{\theta}{2}\right) \\ ie^{-i\beta}\sin\left(\frac{\theta}{2}\right) & e^{-i\alpha}\cos\left(\frac{\theta}{2}\right) \end{bmatrix},\ \theta \in [0,\pi],\ \alpha,\beta \in [0,2\pi)$$

$$U(\theta,\phi) = \begin{bmatrix} e^{i\phi}\cos\left(\frac{\theta}{2}\right) & ie^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ ie^{-i\phi}\sin\left(\frac{\theta}{2}\right) & e^{-i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix},\ \theta \in [0,\pi],\ \phi \in [0,2\pi]$$

Quantum Prisoner's Dilemma realised on EWL

$$C = U(0,0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = U(\pi, 0, \tfrac{\pi}{2}) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$|\psi\rangle = \frac{\sqrt{2}\,(|00\rangle + i|11\rangle)}{2}$$

Expected payoff function

$$|\psi\rangle = J^{\dagger}\left(U_{\mathrm{A}} \otimes U_{\mathrm{B}}\right) J v \; = \begin{bmatrix} -\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_A+\alpha_B\right)+\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\phi_A+\phi_B\right) \\ \sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\phi_B\right)+\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_A}{2}\right)\sin\left(\alpha_B+\phi_A\right) \\ \sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_A+\phi_B\right)+\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_A}{2}\right)\cos\left(\alpha_B+\phi_A\right) \\ -\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\alpha_B\right)+\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\sin\left(\phi_A+\phi_B\right) \end{bmatrix}$$

$$\begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix}$$

$$\$_{\mathrm{B}}(\theta_{\mathrm{A}}, \phi_{\mathrm{A}}, \alpha_{\mathrm{A}}, \theta_{\mathrm{B}}, \phi_{\mathrm{B}}, \alpha_{\mathrm{B}}) =$$

$$3\left(\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_A+\alpha_B\right)-\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\phi_A+\phi_B\right)\right)^2$$
$$+\left(\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\alpha_B\right)-\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\sin\left(\phi_A+\phi_B\right)\right)^2$$
$$+5\left(\sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\phi_B\right)+\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_A}{2}\right)\sin\left(\alpha_B+\phi_A\right)\right)^2$$

EWL library – live demo 1

https://github.com/tomekzaw/ewl/blob/master/examples/example.ipynb

Best response function

$$U_{\mathrm{A}} = U(\mathbf{x}_{\mathrm{A}}) \;\text{ and }\; U_{\mathrm{B}} = U(\mathbf{x}_{\mathrm{B}})$$

$$\text{best response}_{\mathrm{B}}(\mathbf{x}_{\mathrm{A}}) = \underset{\mathbf{x}^*_{\mathrm{B}} \in X}{\operatorname{argmax}} \,\$_{\mathrm{B}}(U(\mathbf{x}_{\mathrm{A}}), U(\mathbf{x}^*_{\mathrm{B}}))$$

$$\text{best response}_{\mathrm{A}}(\mathbf{x}_{\mathrm{B}}) = \underset{\mathbf{x}^*_{\mathrm{A}} \in X}{\operatorname{argmax}} \,\$_{\mathrm{A}}(U(\mathbf{x}^*_{\mathrm{A}}), U(\mathbf{x}_{\mathrm{B}}))$$

Nash equilibrium

In a Nash equilibrium, each player is assumed to know the equilibrium strategies of the other players, and no one has anything to gain by changing only one's own strategy.

$$\mathbf{x}_{\mathrm{A}} = \text{best response}_{\mathrm{A}}(\text{best response}_{\mathrm{B}}(\mathbf{x}_{\mathrm{A}}))$$

$$\mathbf{x}_{\mathrm{A}} = f(\mathbf{x}_{\mathrm{A}})) \quad \text{where } f = (\text{best response}_{\mathrm{A}} \circ \text{best response}_{\mathrm{B}})$$

$$\mathbf{x}_{\mathrm{A}} - f(\mathbf{x}_{\mathrm{A}})) = \mathbf{0}$$

Algorithm

1. Find best response function – argmax of payoff function
2. Find Nash equilibria – fixed points of best response function

Symbolic parametric optimization

In[1]:= **Maximize[a x^2 + b x + c, x]**

Out[1]=

$$\left\{\begin{cases} c & (b == 0 \,\&\&\, a == 0) \,||\, (b == 0 \,\&\&\, a < 0) \\ -\frac{b^2-4ac}{4a} & (b > 0 \,\&\&\, a < 0) \,||\, (b < 0 \,\&\&\, a < 0) \\ \infty & \text{True} \end{cases},\right.$$

$$\left.\left\{x \to \begin{cases} 0 & (b == 0 \,\&\&\, a == 0) \,||\, (b == 0 \,\&\&\, a < 0) \\ -\frac{b}{2a} & (b > 0 \,\&\&\, a < 0) \,||\, (b < 0 \,\&\&\, a < 0) \\ \text{Indeterminate} & \text{True} \end{cases}\right\}\right\}$$

SymPy

WOLFRAM MATHEMATICA 13

Best response function – symbolic approach

$$3\left(\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_A+\alpha_B\right)-\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\phi_A+\phi_B\right)\right)^2$$
$$+\left(\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\alpha_B\right)-\sin\left(\phi_A+\phi_B\right)\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\right)^2$$
$$+5\left(\sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A+\phi_B\right)+\sin\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_B+\phi_A\right)\cos\left(\frac{\theta_A}{2}\right)\right)^2=5$$

Best response function – symbolic approach

$$\left(\sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos\left(\alpha_A + \phi_B\right) + \sin\left(\frac{\theta_B}{2}\right)\sin\left(\alpha_B + \phi_A\right)\cos\left(\frac{\theta_A}{2}\right)\right)^2 = 1$$

Best response function – symbolic approach

$$\begin{bmatrix} -\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin(\alpha_A+\alpha_B)+\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)\cos(\phi_A+\phi_B) \\ \sin\left(\frac{\theta_A}{2}\right)\sin(\alpha_A+\phi_B)\cos\left(\frac{\theta_B}{2}\right)+\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_A}{2}\right)\cos(\alpha_B+\phi_A) \\ -\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\cos(\alpha_A+\alpha_B)+\sin(\phi_A+\phi_B)\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Best response function – numerical approach

	method	start	bounds	success_rate
0	Powell	zero	True	8694
1	Powell	zero	False	9994
2	Powell	random	True	7634
3	Powell	random	False	9923
4	Powell	alice	True	7914
5	Powell	alice	False	9935
6	Nelder-Mead	zero	True	4589
7	Nelder-Mead	zero	False	9660
8	Nelder-Mead	random	True	4147
9	Nelder-Mead	random	False	9952
10	Nelder-Mead	alice	True	4355
11	Nelder-Mead	alice	False	9901

Nash equilibrium search – numerical approach

	player	theta_num	phi_num	alpha_num	payoff
0	Alice	2.668878	1.629450	2.925350	NaN
1	Bob	-0.473305	0.215931	-0.059275	4.999999
2	Alice	-2.668066	0.059188	1.354862	5.000000
3	Bob	5.809687	1.786669	-1.630399	5.000000
4	Alice	2.669466	-1.507399	-0.215242	4.999995
5	Bob	-0.472682	0.214946	-0.063987	4.999999
6	Alice	-2.668677	0.063894	1.355847	5.000000
7	Bob	5.810297	1.785683	-1.635118	5.000000
8	Alice	2.670080	-1.502658	-0.214255	4.999995
9	Bob	-0.472033	0.213973	-0.068695	4.999999
10	Alice	-2.669315	0.068598	1.356819	5.000000
11	Bob	5.810932	1.784706	-1.639856	5.000000
12	Alice	2.670716	-1.497901	-0.213277	4.999995
13	Bob	-0.471359	0.213011	-0.073419	5.000000
14	Alice	-2.669977	0.073316	1.357781	5.000000
15	Bob	5.811602	1.783718	-1.644761	5.000000
16	Alice	2.671389	-1.492974	-0.212288	4.999995
17	Bob	-0.470648	0.212038	-0.078311	5.000000
18	Alice	-2.670675	0.078202	1.358753	5.000000
19	Bob	5.812289	1.782752	-1.649607	5.000000
20	Alice	2.672078	-1.488108	-0.211321	4.999995

Best response cycle

$$\widehat{A} = \hat{U}(\theta_A, \phi_A, \alpha_A)$$

$$\hat{B} = \hat{U}(\theta_A + \pi, \alpha_A + \frac{\pi}{2}, -\phi_A)$$

$$\widehat{A}' = \hat{U}(\theta_A, \phi_A - \frac{\pi}{2}, \alpha_A + \frac{\pi}{2})$$

$$\hat{B}' = \hat{U}(\theta_A + \pi, -\alpha_A, \phi_A + \frac{\pi}{2})$$

Mixed quantum strategies

$$S_A(\gamma_A) = \cos^2\frac{\gamma_A}{2}\widehat{A} + \sin^2\frac{\gamma_A}{2}\widehat{A}'; \quad \gamma_A \in [0,\pi]$$

$$S_B(\gamma_B) = \cos^2\frac{\gamma_B}{2}\hat{B} + \sin^2\frac{\gamma_B}{2}\hat{B}'; \quad \gamma_B \in [0,\pi]$$

$$\rho = |\varphi\rangle\langle\varphi| \qquad C(\rho) = pU_1\rho U_1^\dagger + (1-p)U_2\rho U_2^\dagger$$

EWL library – live demo 2

https://github.com/tomekzaw/ewl/blob/master/examples/mixed_strategies.py

Performance tests

$$\frac{1}{\sqrt{2}}(|0...0\rangle + i\,|1...1\rangle)$$

qubits count	execution time [s]
2 qubits	2,55
2 qubits	2,14
2 qubits	2,14
3 qubits	447
3 qubits	437
3 qubits	446

$$\frac{1}{2}(|0...0\rangle + \sqrt{3}\,|1...1\rangle)$$

qubits count	execution time [s]
2 qubits	37,9
2 qubits	35
2 qubits	36,2
3 qubits	2h 57min 17s
3 qubits	3h 15min 46s
3 qubits	2h 54min 4s

$$|0...0\rangle$$

qubits count	execution time [s]
2 qubits	1,21
2 qubits	1,04
2 qubits	1,12
3 qubits	10,6
3 qubits	10,8
3 qubits	10,9
4 qubits	126
4 qubits	120
4 qubits	122
5 qubits	2701 (45min 1s)
5 qubits	2692 (44min 52s)
5 qubits	2743 (45min 43s)

References

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[4] Tomasz Zawadzki and Piotr Kotara. “A Python tool for symbolic analysis of quantum games in EWL protocol with IBM Q integration.” https://github.com/tomekzaw/ewl/