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Tensor Networks approach to simulating Continuous-Time Stochastic Automata Networks

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Agenda

Motivation

- Probabilistic models in Operations Research and Performance Evaluation are mostly focused around Markovian ones.
- Simulation capabilities (considering both memory and time) are limited due to the ubiquitous state space explosion problem.
- The Stochastic Automata Networks formalism utilizes a hierarchical representation of distributed systems in order to overcome the aforementioned limitations, however there is still no efficient numerical algorithm proposed for a notable class of models.
- On the other hand, the Tensor Networks formalism proved to be helpful in dealing with complex, many-body quantum systems struggling with the curse of dimensionality problem.

Stochastic models

Definition

The *stochastic process* is given by a set $\left\{\chi_t : t \in \mathbb{R}^{0^+}\right\}$ of random variables, each of which taking values from a set $\mathscr{S} = \left\{s^{(1)}, s^{(2)}, \ldots, s^{(N)}\right\}$ called the *state space*.

Remark

A stochastic process is said to be *Markovian* when it satisfies the *memorylessness property* stating that the future state of the process depends upon the present state only and not on the sequence of preceding events.

Stochastic models

		State Space	
		Discrete	Continuous
Time Space	Discrete	DTMC	DTMP
	Continuous	**CTMC**	CTMP

Table: The family of Markov models.

Stochastic models

Definition

Consider a stochastic process $\mathscr{X} = \left\{\chi_t : t \in \mathbb{R}^{0^+}\right\}$ over a state space $\mathscr{S}$. Then, for any value $k \in \mathbb{N}$, strictly increasing times indexed up to these values $t_0 < t_1 < t_2 < \ldots \in \mathbb{R}^{0^+}$ and all states indexed at these times $s_0, s_1, s_2, \ldots \in \mathscr{S}$, the stochastic process $\mathscr{X}$ constitutes the *Continuous-Time Markov Chain* if it satisfies the following *Markov property*:

$$\Pr\left(\chi_{t_{k+1}} = s_{k+1} \mid \chi_{t_k} = s_k, \chi_{t_{k-1}} = s_{k-1}, \ldots, \chi_{t_0} = s_0\right) = \Pr\left(\chi_{t_{k+1}} = s_{k+1} \mid \chi_{t_k} = s_k\right),$$

where: $\Pr\left(\cdot \mid \cdot\right)$ denotes the conditional probability.

Stochastic models

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Definition

A CTMC $\mathscr{X} = \left\{\chi_t : t \in \mathbb{R}^{0^+}\right\}$ over a state space $\mathscr{S}$ is said to be *time-homogeneous* if its conditional transition probability is invariant with respect to time, i.e.:

$$\Pr\left(\chi_t = s \mid \chi_{t'} = s'\right) = \Pr\left(\chi_{t-t'} = s \mid \chi_0 = s'\right),$$

where: $t' \leq t \in \mathbb{R}^{0+}$ and $s, s' \in \mathscr{S}$.

Remark

The aforementioned transition probabilities form a *probability matrix* $P(t)_{N\times N}$ describing evolution of a CTMC, such that:

$$p_{ij}(t) = \Pr\left(\chi_t = s^{(j)} \mid \chi_0 = s^{(i)}\right).$$

Stochastic models

Theorem

The probability matrix $P(t)$ satisfies both the *Kolmogorov forward equation*:

$$\frac{d}{dt}P(t) = P(t)Q,$$

and the *Kolmogorov backward equation*:

$$\frac{d}{dt}P(t) = QP(t),$$

for a transition rate matrix $Q_{N\times N}$ with elements:

$$q_{ij} = \lim_{\Delta t\to 0} \frac{p_{ij}(\Delta t) - p_{ij}(0)}{\Delta t} \quad \text{for } j \neq i, \quad q_{ii} = -\sum_{j\neq i} q_{ij}.$$

Stochastic models

Theorem

Taking an initial condition $P(0) = I_N$, the unique solution of Kolmogorov equations is given by:

$$P(t) = \exp(tQ).$$

Remark

Let $\mathbf{p}(t)$ be a probability distribution of states belonging to a time-homogeneous CTMC $\mathscr{X}$ over time. Assuming $\mathbf{p}(0)$ is the initial probability distribution, the probability distribution of states of $\mathscr{X}$ at any instant of time t is given by:

$$\mathbf{p}(t) = \mathbf{p}(0)P(t).$$

Stochastic Automata

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Definition

The *Continuous-Time Stochastic Automaton* $\mathcal{A}$ is a triple:

- $\mathscr{S}_{\mathcal{A}} = \left\{s^{(i)}\right\}$ of size $S_{\mathcal{A}}$,
- $\mathscr{L}_{\mathcal{A}} = \left\{\left(e, p^{e}_{s^{(i)}}\right)\right\}$ of size $L_{\mathcal{A}}$,
- $f_{\mathcal{A}} : \mathscr{S}^2_{\mathcal{A}} \to \mathscr{L}_{\mathcal{A}}$.

Definition

The (Continuous-Time) *Stochastic Automata Network* $\mathcal{N}$ is a tuple:

- $\mathscr{A}_{\mathcal{N}} = \left\{\mathcal{A}^{(i)}\right\}$ of size $N_{\mathcal{N}}$,
- $\mathscr{E}_{\mathcal{N}} = \left\{e^{(i)}\right\}$ of size $E_{\mathcal{N}}$.

Stochastic Automata

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Definition

Each event $e \in \mathscr{E}_{\mathcal{N}}$ is given by a triple:

- $t_{\mathcal{N}}(e)$ for $t_{\mathcal{N}} : \mathscr{E}_{\mathcal{N}} \to \{loc, syn\}$,
- $m_{\mathcal{N}}(e)$ for $m_{\mathcal{N}} : \mathscr{E}_{\mathcal{N}} \to \mathscr{A}_{\mathcal{N}}$,
- $q_{\mathcal{N}}(e)$ for $q_{\mathcal{N}} : \mathscr{E}_{\mathcal{N}} \to \mathbb{R}^{0^+}$.

Stochastic Automata

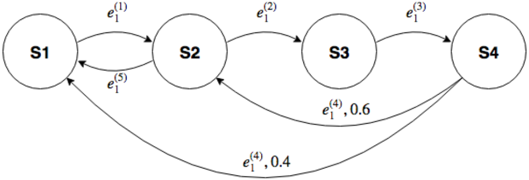

Figure: Graph representation of a stochastic automaton.

Event	$e^{(1)}$	$e^{(2)}$	$e^{(3)}$	$e^{(4)}$	$e^{(5)}$
Rate	λ_1	λ_2	λ_3	λ_4	λ_5

Table: Transition rates of events associated with the automaton above.

Stochastic Automata

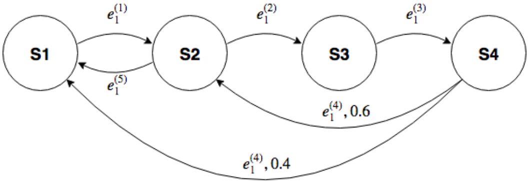

Figure: Graph representation of a stochastic automaton.

$$Q_{\mathcal{A}} = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ \lambda_5 & -(\lambda_2 + \lambda_5) & \lambda_2 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 \\ 0.4 \cdot \lambda_4 & 0.6 \cdot \lambda_4 & 0 & -\lambda_4 \end{bmatrix}$$

Stochastic Automata

Definition

Let $A_{m\times n} = [a_{ij}]$ and $B_{p\times q} = [b_{ij}]$. The *Kronecker* (also called *tensor*) *product* is the block matrix $(A \otimes B)_{mp\times nq}$, such that:

$$A \otimes B \stackrel{\text{def}}{=} \begin{bmatrix} a_{11}b_{11} & \dots & a_{11}b_{1q} & \dots & \dots & a_{1n}b_{11} & \dots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots & \dots & \dots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & \dots & a_{11}b_{pq} & \dots & \dots & a_{1n}b_{p1} & \dots & a_{1n}b_{pq} \\ \vdots & & \vdots & \ddots & & \vdots & & \vdots \\ \vdots & & \vdots & & \ddots & \vdots & & \vdots \\ a_{m1}b_{11} & \dots & a_{m1}b_{1q} & \dots & \dots & a_{mn}b_{11} & \dots & a_{mn}b_{1q} \\ \vdots & \ddots & \vdots & \dots & \dots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & \dots & a_{m1}b_{pq} & \dots & \dots & a_{mn}b_{p1} & \dots & a_{mn}b_{pq} \end{bmatrix}.$$

Stochastic Automata

Definition

Let $A_{n\times n}$ and $B_{m\times m}$. The *Kronecker* (also called *tensor*) *sum* is the matrix $(A \oplus B)_{mn\times mn}$ such that:

$$A \oplus B \stackrel{\text{def}}{=} A \otimes I_m + I_n \otimes B.$$

Remark

The generalized operations:

$$\bigotimes_{k=1}^{N} A^{(k)} = A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(N)} \quad \text{and} \quad \bigoplus_{k=1}^{N} B^{(k)} = B^{(1)} \oplus B^{(2)} \oplus \ldots \oplus B^{(N)}$$

are well defined for any rectangular matrices $A^{(1)}, A^{(2)}, \ldots, A^{(N)}$ and any square matrices $B^{(1)}, B^{(2)}, \ldots, B^{(N)}$.

Stochastic Automata

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Theorem

The global infinitesimal generator matrix $Q_{\mathcal{N}}$ of a Stochastic Automata Network $\mathcal{N}$ may be written in terms of $N_{\mathcal{N}} + 2 \cdot E_{\mathcal{N}}$ tensor products by separating local and synchronized transitions as follows:

$$Q_{\mathcal{N}} = \bigoplus_{k=1}^{N_{\mathcal{N}}} Q^{loc}_{\mathcal{A}^{(k)}} + \sum_{e \in \mathscr{E}^{syn}_{\mathcal{N}}} \left[\bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{pos}}_{\mathcal{A}^{(k)}} + \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{neg}}_{\mathcal{A}^{(k)}} \right].$$

Stochastic Automata

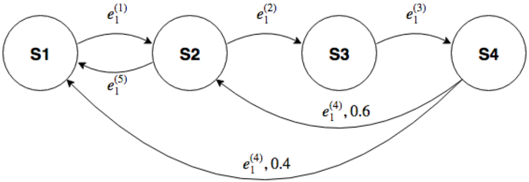

Figure: Graph representation of a stochastic automaton.

$$Q_{\mathcal{A}}^{loc} = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ \lambda_5 & -\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.4 \cdot \lambda_4 & 0.6 \cdot \lambda_4 & 0 & -\lambda_4 \end{bmatrix}$$

Stochastic Automata

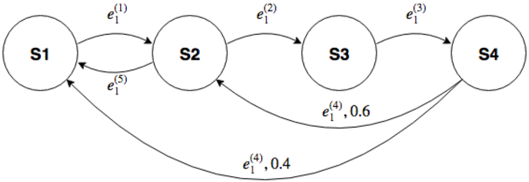

Figure: Graph representation of a stochastic automaton.

$$Q_{\mathcal{A}}^{\left(e_1^{(2)}\right)_{pos}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad Q_{\mathcal{A}}^{\left(e_1^{(2)}\right)_{neg}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Stochastic Automata

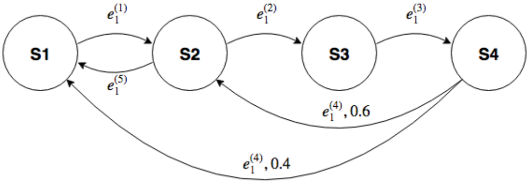

Figure: Graph representation of a stochastic automaton.

$$Q_{\mathcal{A}}^{\left(e_1^{(3)}\right)_{pos}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q_{\mathcal{A}}^{\left(e_1^{(3)}\right)_{neg}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Tensors

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Definition

The *rank-d tensor* $T_{\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(d)}}$ of *indices* $\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(d)}$, each of which of *size* $\left|\alpha^{(i)}\right|$, $1 \le i \le d$, is an element of the $\mathbb{R}^{\left|\alpha^{(1)}\right|\cdot\left|\alpha^{(2)}\right|\cdot\ldots\cdot\left|\alpha^{(d)}\right|}$ space.

Remark

An element of the tensor $T_{\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(d)}}$ is given by $T_{i_1,i_2,\dots,i_d}$ for $1 \le i_1 \le \left|\alpha^{(1)}\right|, 1 \le i_2 \le \left|\alpha^{(2)}\right|, \dots, 1 \le i_d \le \left|\alpha^{(d)}\right|$.

Tensors

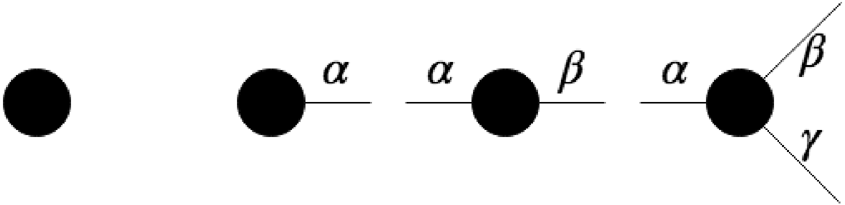

Figure: Diagrammatic notation of tensors.

Tensors

Definition

Let $T^{(1)}_{\alpha_1^{(1)},\alpha_1^{(2)},\dots,\alpha_1^{(d_1)}}$ and $T^{(2)}_{\alpha_2^{(1)},\alpha_2^{(2)},\dots,\alpha_2^{(d_2)}}$ be tensors. The *tensor product* of $T^{(1)}$ and $T^{(2)}$ is a tensor $\left(T^{(1)} \otimes T^{(2)}\right)_{\alpha_1^{(1)},\alpha_1^{(2)},\dots,\alpha_1^{(d_1)},\alpha_2^{(1)},\alpha_2^{(2)},\dots,\alpha_2^{(d_2)}}$, being a result of the element-wise product of the values belonging to each constituent tensor:

$$\left(T^{(1)} \otimes T^{(2)}\right)_{i_1,i_2,\dots,i_{d_1},j_1,j_2,\dots,j_{d_2}} = T^{(1)}_{i_1,i_2,\dots,i_{d_1}} \cdot T^{(2)}_{j_1,j_2,\dots,j_{d_2}},$$

where: $\forall_{1\leq k\leq d_1} : 1 \leq i_k \leq \left|\alpha_1^{(k)}\right|$ and $\forall_{1\leq k\leq d_2} : 1 \leq j_k \leq \left|\alpha_2^{(k)}\right|$.

Tensors

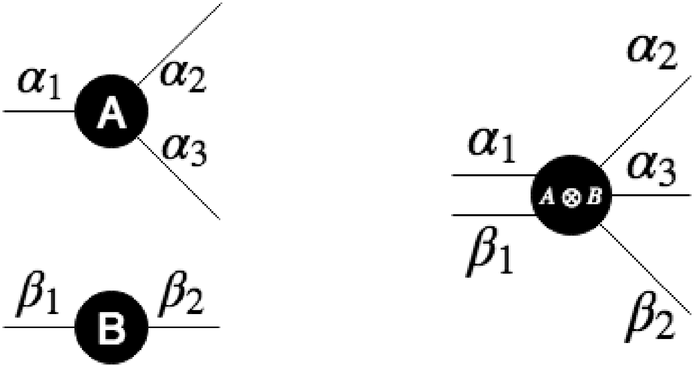

Figure: Diagrammatic notation of a tensor product.

Tensors

Definition

Let $T^{(1)}_{\alpha_1^{(1)},\ldots,\alpha_1^{(m-1)},\alpha,\alpha_1^{(m+1)},\ldots,\alpha_1^{(d_1)}}$ and $T^{(2)}_{\alpha_2^{(1)},\ldots,\alpha_2^{(n-1)},\alpha,\alpha_2^{(n+1)},\ldots,\alpha_2^{(d_2)}}$ be tensors with corresponding indices $\alpha_1^{(m)}$ and $\alpha_2^{(n)}$, $1 \leq m \leq d_1$ and $1 \leq n \leq d_2$, denoted by α. The *contraction* of $T^{(1)}$ and $T^{(2)}$ by α is a tensor $\left(T^{(1)} \circ_\alpha T^{(2)}\right)$, such that:

$$\left(T^{(1)} \circ_\alpha T^{(2)}\right)_{\alpha_1^{(1)},\ldots,\alpha_1^{(m-1)},\alpha_1^{(m+1)},\ldots,\alpha_1^{(d_1)},\alpha_2^{(1)},\ldots,\alpha_2^{(n-1)},\alpha_2^{(n+1)},\ldots,\alpha_2^{(d_2)}} \stackrel{\text{def}}{=} \sum_{k=1}^{|\alpha|} T^{(1)}_{\alpha_1^{(1)},\ldots,\alpha_1^{(m-1)},k,\alpha_1^{(m+1)},\ldots,\alpha_1^{(d_1)}} \cdot T^{(2)}_{\alpha_2^{(1)},\ldots,\alpha_2^{(n-1)},k,\alpha_2^{(n+1)},\ldots,\alpha_2^{(d_2)}}.$$

Tensors

Example

$$C_{\alpha,\gamma} = \sum_{k=1}^{|\beta|} A_{\alpha,k} \cdot B_{k,\gamma}$$

$$c_{ij} = \sum_{k=1}^{|\beta|} a_{ik} \cdot b_{kj}$$

Figure: Diagrammatic notation of a rank-two tensors contraction.

Tensors

Definition

Let $T_{\alpha(1),\alpha(2),\ldots,\alpha(d)}$ be a rank-d tensor. The *Tensor Train* decomposition of the tensor T is given by the sequence of d tensors $G^{(1)}_{\beta(0),\alpha(1),\beta(1)}, G^{(2)}_{\beta(1),\alpha(2),\beta(2)}, \ldots, G^{(d)}_{\beta(d-1),\alpha(d),\beta(d)}$, such that:

$$T_{\alpha(1),\alpha(2),\ldots,\alpha(d)} = G^{(1)}_{\beta(0),\alpha(1),\beta(1)} \circ_{\beta(1)} G^{(2)}_{\beta(1),\alpha(2),\beta(2)} \circ_{\beta(2)} \cdots \circ_{\beta(d-1)} G^{(d)}_{\beta(d-1),\alpha(d),\beta(d)}.$$

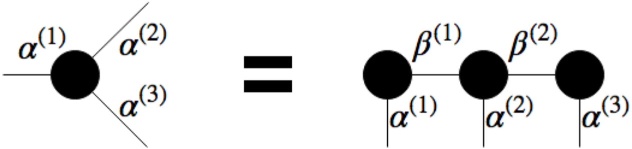

Figure: Diagrammatic notation of a rank-three tensor TT decomposition.

Tensor Networks

Definition

A set $\mathscr{T} = \left\{ T^{(1)}_{\alpha_1^{(1)},\alpha_1^{(2)},\dots,\alpha_1^{(d_1)}}, T^{(2)}_{\alpha_2^{(1)},\alpha_2^{(2)},\dots,\alpha_2^{(d_2)}}, \dots, T^{(N)}_{\alpha_N^{(1)},\alpha_N^{(2)},\dots,\alpha_N^{(d_N)}} \right\}$ of N tensors, where some – or all – of their indices are subjected to contraction, is called the *Tensor Network*.

A note on matrix exponential

Definition

Let $A_{n\times n}$. The *exponential of* A is the $\exp(A)_{n\times n}$ matrix given by the following infinite power series:

$$\exp(A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots,$$

where: $k!$ is the factorial of k.

Definition

Let $A_{n\times n}$ and $B_{n\times n}$. The *commutator* of matrices A and B is the matrix $[A, B]_{n\times n}$ such that:

$$[A, B] \stackrel{\text{def}}{=} AB - BA.$$

A note on matrix exponential

Theorem

For any two commutative matrices $A_{n\times n}$ and $B_{n\times n}$, the exponential of their sum may be expressed in terms of a product of their exponentials:

$$\exp(A+B) = \exp(B+A) = \exp(A)\exp(B) = \exp(B)\exp(A).$$

Theorem

Let $A_{n\times n}$ and $B_{n\times n}$. The *Lie product formula* (also called *Trotter decomposition* or *Suzuki-Trotter expansion of the first order*) states that:

$$\exp(A+B) = \lim_{k\to\infty}\left(\exp\left(\frac{1}{k}A\right)\exp\left(\frac{1}{k}B\right)\right)^k.$$

TNSAN derivation

Recalling the solution of Kolmogorov equations:

$$P(t) = \exp\left(t \cdot Q\right),$$

and the definition of SAN descriptor:

$$Q_{\mathcal{N}} = \bigoplus_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{loc} + \sum_{e \in \mathcal{E}_{\mathcal{N}}^{syn}} \left[\bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{pos}} + \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{neg}} \right],$$

the following formula holds:

$$P(t) = \exp\left(t \cdot \bigoplus_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{loc} + \sum_{e \in \mathcal{E}_{\mathcal{N}}^{syn}} \left[t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{pos}} + t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{neg}} \right] \right).$$

TNSAN derivation

$$P(t) = \exp\left(t \cdot \bigoplus_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{loc} + \sum_{e \in \mathscr{E}_{\mathcal{N}}^{syn}} \left[t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{pos}} + t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{neg}} \right] \right).$$

Applying the Suzuki-Trotter expansion:

$$\exp(A + B) = \lim_{k \to \infty} \left(\exp\left(\frac{1}{k} A\right) \exp\left(\frac{1}{k} B\right) \right)^k,$$

and denoting $\Delta t = \frac{t}{n}$, one may obtain:

$$P(t) = \lim_{n \to \infty} \left(\exp\left(\Delta t \cdot \bigoplus_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{loc} \right) \cdot \prod_{e \in \mathscr{E}_{\mathcal{N}}^{syn}} \left[\exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{pos}} \right) \cdot \exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q_{\mathcal{A}^{(k)}}^{e_{neg}} \right) \right] \right)^n.$$

TNSAN derivation

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Theorem

Let $A_{n\times n}$ and $B_{m\times m}$. The exponential of the Kronecker sum of these matrices may be expressed in terms of the Kronecker product of their exponentials:

$$\exp(A \oplus B) = \exp(A) \otimes \exp(B).$$

TNSAN derivation

$$P(t) = \lim_{n\to\infty} \left(\exp\left(\Delta t \cdot \bigoplus_{k=1}^{N_{\mathcal{N}}} Q^{loc}_{\mathcal{A}^{(k)}}\right) \cdot \prod_{e\in\mathscr{E}^{syn}_{\mathcal{N}}} \left[\exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{pos}}_{\mathcal{A}^{(k)}}\right) \cdot \exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{neg}}_{\mathcal{A}^{(k)}}\right)\right]\right)^n .$$

Employing the aforementioned theorem:

$$\exp(A \oplus B) = \exp(A) \otimes \exp(B),$$

and for sufficiently large $n \in \mathbb{N}$, it holds:

$$P(t) \cong \left(\bigotimes_{k=1}^{N_{\mathcal{N}}} \exp\left(\Delta t \cdot Q^{loc}_{\mathcal{A}^{(k)}}\right) \cdot \prod_{e\in\mathscr{E}^{syn}_{\mathcal{N}}} \left[\exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{pos}}_{\mathcal{A}^{(k)}}\right) \cdot \exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{neg}}_{\mathcal{A}^{(k)}}\right)\right]\right)^n .$$

TNSAN derivation

$$P(t) \cong \left(\bigotimes_{k=1}^{N_{\mathcal{N}}} \exp\left(\Delta t \cdot Q^{loc}_{\mathcal{A}^{(k)}}\right) \cdot \prod_{e \in \mathscr{E}^{syn}_{\mathcal{N}}} \left[\exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{pos}}_{\mathcal{A}^{(k)}} \right) \cdot \exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{neg}}_{\mathcal{A}^{(k)}} \right) \right] \right)^n .$$

Finally, each factor of the preceding product for two automata $\mathcal{A}^{(i)}, \mathcal{A}^{(j)}$ affected by a synchronizing event e may be expressed as:

$$\left(\bigotimes_{k=1}^{i-1} I_{S_{\mathcal{A}^{(k)}}} \right) \otimes R \cdot \left[\left(\bigotimes_{k=i+1}^{j-1} I_{S_{\mathcal{A}^{(k)}}} \right) \otimes \exp\left(\Delta t \cdot Q^{e_{pos}}_{\mathcal{A}^{(j)}} \otimes Q^{e_{pos}}_{\mathcal{A}^{(i)}} \right) \right] \cdot$$

$$\left[\left(\bigotimes_{k=i+1}^{j-1} I_{S_{\mathcal{A}^{(k)}}} \right) \otimes \exp\left(\Delta t \cdot Q^{e_{neg}}_{\mathcal{A}^{(j)}} \otimes Q^{e_{neg}}_{\mathcal{A}^{(i)}} \right) \right] \cdot R^{-1} \otimes \left(\bigotimes_{k=j+1}^{N_{\mathcal{N}}} I_{S_{\mathcal{A}^{(k)}}} \right) .$$

Pseudocode

procedure TNSAN($Q_{\mathcal{N}}$, $\mathbf{p}_{\mathcal{N}}(0)$, t, n):

  $\Delta t := \frac{t}{n}$

  if $\mathbf{p}_{\mathcal{N}}(0)$ **is** joint probability distribution:

    $\mathbb{S}^{0}_{\boldsymbol{\mathcal{N}}} :=$ TensorTrain($\mathbf{p}_{\mathcal{N}}(0)$)

  else:

    $\mathbb{S}^{0}_{\boldsymbol{\mathcal{N}}} := \mathbf{p}_{\mathcal{N}}(0)$

Pseudocode

$$\mathbb{E}^{\Delta t}_{\mathcal{N}} := \bigotimes_{k=1}^{N_{\mathcal{N}}} \exp\left(\Delta t \cdot Q^{loc}_{\mathcal{A}(k)}\right) \cdot \prod_{e \in \mathscr{E}^{syn}_{\mathcal{N}}} \left[\exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{pos}}_{\mathcal{A}(k)}\right) \cdot \exp\left(\Delta t \cdot \bigotimes_{k=1}^{N_{\mathcal{N}}} Q^{e_{neg}}_{\mathcal{A}(k)}\right)\right]$$

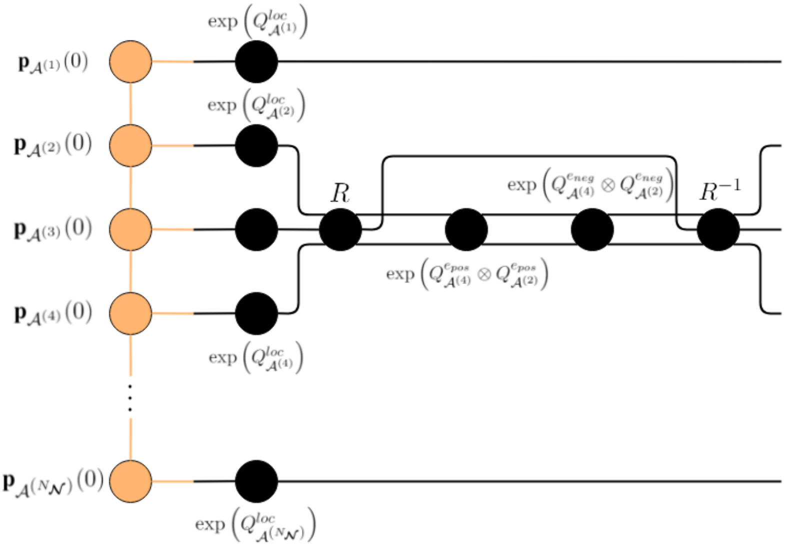
for k **in** 1 .. n:

$$\mathbb{S}^{k \cdot \Delta t}_{\mathcal{N}} := \mathbb{S}^{(k-1)\cdot \Delta t}_{\mathcal{N}} \circ \mathbb{E}^{\Delta t}_{\mathcal{N}}$$

return $\mathbb{S}^{t}_{\mathcal{N}}$

end

TNSAN Tensor Network



Benchmark problem

- The TNSAN algorithm has been implemented with Julia programming language.
- Multiple numerical tests have been conducted on the resource sharing mechanism model.
- Obtained results have been compared with the reference ones, determined with the EXPOKIT package.

Numerical results

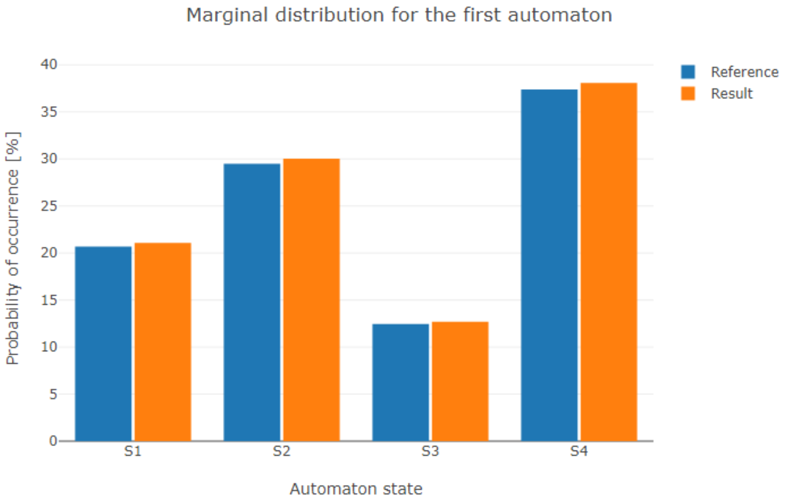


Figure: Marginal distribution of an automaton after 10^5 steps.

Numerical results

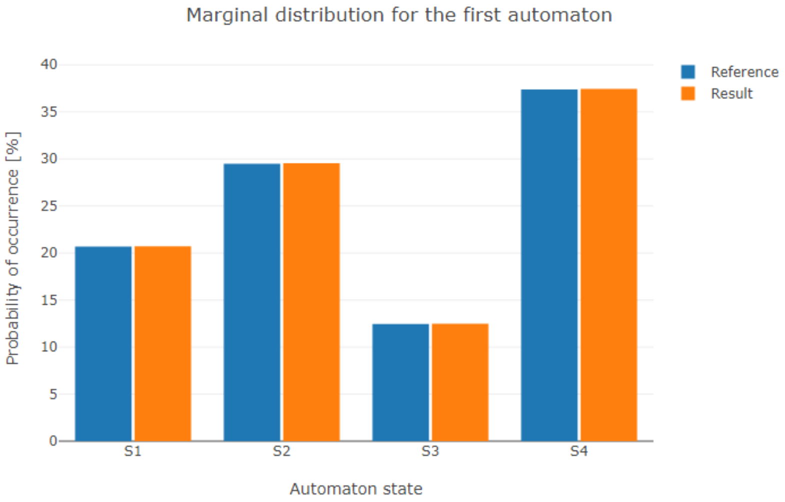


Figure: Marginal distribution of an automaton after 10^6 steps.

Numerical results

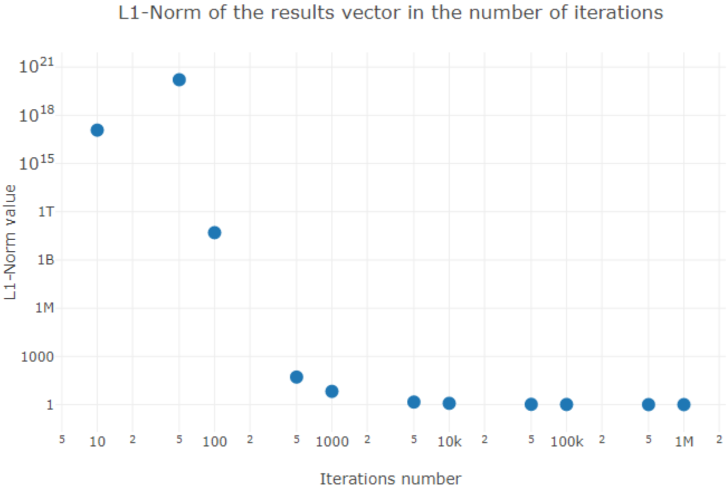


Figure: $\mathscr{L}_1$ of the results vector in the number of iterations.

Numerical results

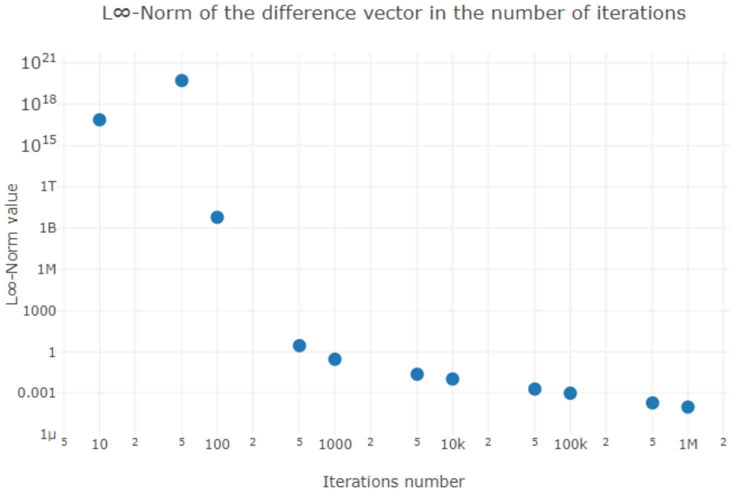


Figure: $\mathscr{L}_\infty$ norm of the difference vector in the number of iterations.

Numerical results

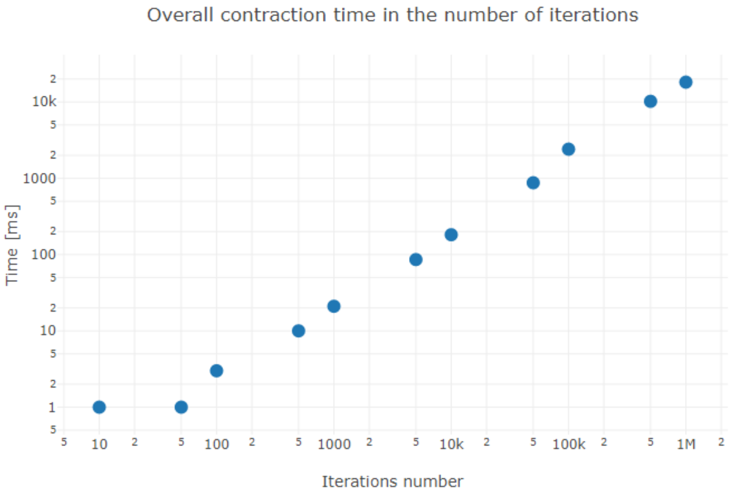


Figure: Main-loop execution time in the number of iterations.

Numerical results

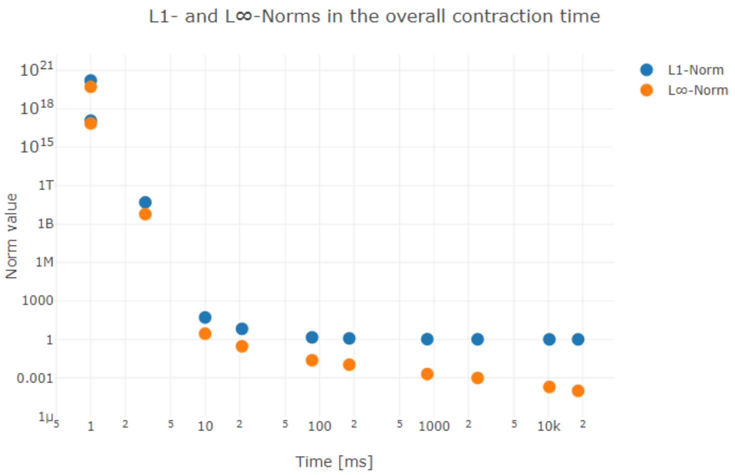


Figure: $\mathscr{L}_1$ and $\mathscr{L}_\infty$ norms in the main-loop execution time.

Conclusions

AGH

- Within this thesis, a novel algorithm for computing the exponential of matrices expressed by a sum of Kronecker products is proposed. Then, a problem of determining transient probability distribution over SAN states is reduced to the matter of contraction between TNs.
- Proposed approach is extensively analyzed, both theoretically and numerically. Furthermore, proper applicability areas for the TNSAN algorithm are pointed out on the basis of comprehensive insight into method's properties.
- This thesis sheds light on previously unexplored field of SANs and TNs hybridization.