AGH University of Science and Technology

Quantum walks in image segmentation

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Presentaion based on the MSc thesis supervised by Katarzyna Rycerz (DCS AGH) and Piotr Gawron (IITIS PAN, Gliwice), Department of Computer Science AGH, Krakow, June 2018.

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Outline

“A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats the process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from the starting point O.”

Karl Pearson, 1905

Definition

Classical random walk is a stochastic process, during which a particle (walker) explores the space by randomly jumping from the current position to a neighboring one, based solely on the current state and according to the transition probabilities given at that state.

- Let n be the size of the position space, then $\vec{p}_t \in \mathbb{R}^n$ denotes a **probability distribution** over position space of finding the walker at given position after t steps.

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- Let n be the size of the position space, then $M_{ij} \in \mathbb{R}^{n \times n}$ is a **transition matrix** determining the probability of movement of the walker from position i to position j in a single step. M satisfies following constraints:

$$0 \leq M_{ij} \leq 1, \quad \forall i, j \in \{0, 1, 2, ..., n-1\},$$
$$\sum_i M_{ij} = 1, \quad \forall j \in \{0, 1, 2, ..., n-1\}.$$

Space-based classification

discrete space random walk

continuous space random walk

Space-based classification

discrete space random walk

continuous space random walk

Time-based classification

discrete time random walk

$$\vec{p}_t = M\vec{p}_{t-1},$$

$$\vec{p}_t = M^t\vec{p}_0.$$

Space-based classification

discrete space random walk

continuous space random walk

Time-based classification

discrete time random walk

$$\vec{p}_t = M\vec{p}_{t-1},$$

$$\vec{p}_t = M^t\vec{p}_0.$$

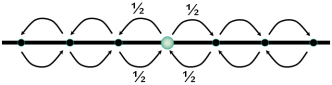
continuous time random walk

$$\frac{d\vec{p}(t)}{dt} = \gamma M\vec{p}(t-1).$$

$$\vec{p}(t) = e^{\gamma Mt}\vec{p}(0).$$

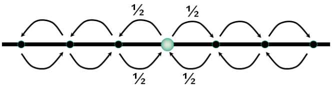
Random walks

Example – discrete time walk on an infinite line



Random walks

Example – discrete time walk on an infinite line



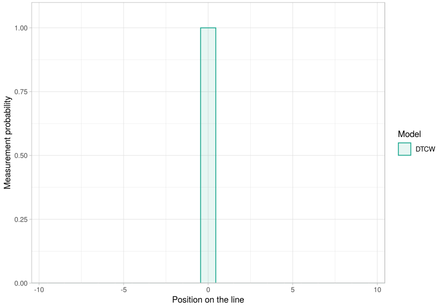


Figure: Probability distribution of measuring walker at given position after 0 steps of discrete time random walk

Random walks

Example – discrete time walk on an infinite line

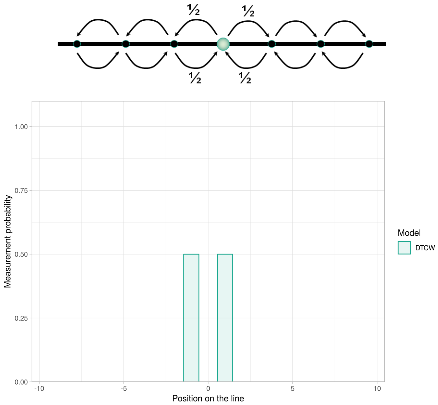
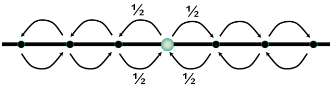


Figure: Probability distribution of measuring walker at given position after 1 step of discrete time random walk

Random walks

Example – discrete time walk on an infinite line



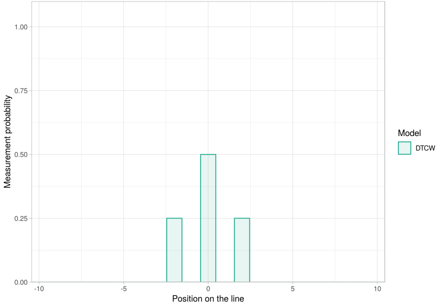
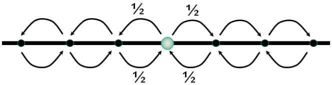


Figure: Probability distribution of measuring walker at given position after 2 steps of discrete time random walk

Random walks

Example – discrete time walk on an infinite line



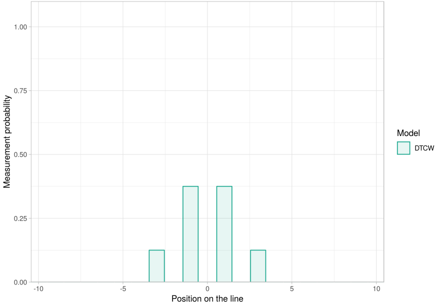


Figure: Probability distribution of measuring walker at given position after 3 steps of discrete time random walk

Random walks

Example – discrete time walk on an infinite line

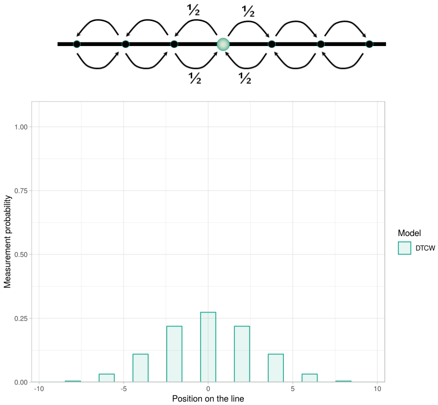
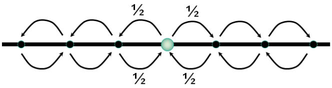


Figure: Probability distribution of measuring walker at given position after 8 steps of discrete time random walk

Random walks

Example – discrete time walk on an infinite line



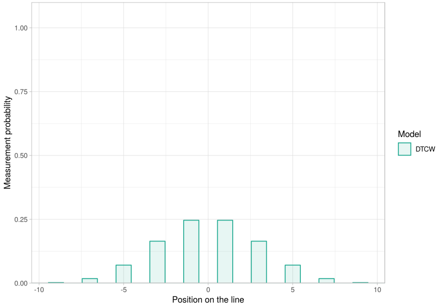
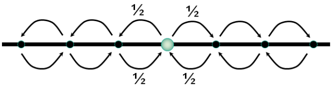


Figure: Probability distribution of measuring walker at given position after 9 steps of discrete time random walk

Random walks

Example – discrete time walk on an infinite line



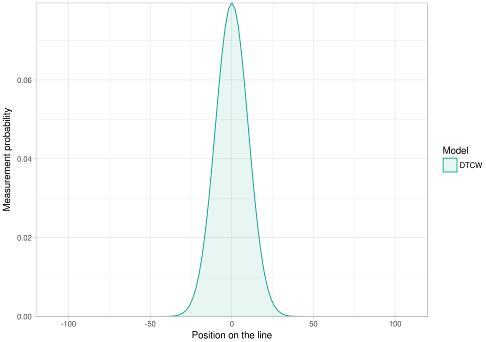


Figure: Probability distribution of measuring walker at given position after 100 steps of discrete time random walk

- determining some properties of given state space
  (hitting time, mixing rate, state connectivity)
- simulation of behavior of entities existing in given state space
- random sampling

- physics (Brownian motion, gas diffusion)
- economy (share prices)
- mathematics (equation solving)
- computer science (Internet size estimation)

- First work on *quantum walks* was published by Aharonov in 1993.
- Quantum walks can benefit from some properties of quantum world and provide polynomial speedup.
- There is similar classification of quantum walks, as in the classical case.
- The behaviour of quantum walks is very much different from its classical counterparts and strongly depends on the choice of evolution operators as well as initial setup.

Position space, $\mathcal{H}_p$

- spanned by the basis vectors corresponding to every achievable walker position
- example basis:

$$\{\ldots, |-2\rangle_p, |-1\rangle_p, |0\rangle_p, |1\rangle_p, |2\rangle_p, \ldots\}$$
$$\{\ldots, |-1,-1\rangle_p, |-1,0\rangle_p, |-1,1\rangle_p, \ldots, |0,-1\rangle_p,$$
$$|0,0\rangle_p, |0,1\rangle_p, \ldots, |1,-1\rangle_p, |1,0\rangle_p, |1,1\rangle_p, \ldots\}$$

- $|\psi\rangle_p = \sum_i a_i|i\rangle_p,\ \sum_i \|a_i\|^2 = 1$

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$$\{\ldots, |-1,-1\rangle_p, |-1,0\rangle_p, |-1,1\rangle_p, \ldots, |0,-1\rangle_p, |0,0\rangle_p, |0,1\rangle_p, \ldots, |1,-1\rangle_p, |1,0\rangle_p, |1,1\rangle_p, \ldots\}$$

- $|\psi\rangle_p = \sum_i a_i |i\rangle_p,\ \sum_i \|a_i\|^2 = 1$

Coin space, $\mathcal{H}_c$

- determines the direction of walker movement
- example basis:

$$\{|\leftarrow\rangle_c, |\rightarrow\rangle_c\}$$
$$\{|\leftarrow\rangle_c, |\rightarrow\rangle_c, |\uparrow\rangle_c, |\downarrow\rangle_c\}$$

- $|\psi\rangle_c = \sum_j b_j |j\rangle_c,\ \sum_j \|b_j\|^2 = 1$

Position space, $\mathcal{H}_p$

- spanned by the basis vectors corresponding to every achievable walker position
- example basis:

$$\{\ldots, |-2\rangle_p, |-1\rangle_p, |0\rangle_p, |1\rangle_p, |2\rangle_p, \ldots\}$$
$$\{\ldots, |-1,-1\rangle_p, |-1,0\rangle_p, |-1,1\rangle_p, \ldots, |0,-1\rangle_p, |0,0\rangle_p, |0,1\rangle_p, \ldots, |1,-1\rangle_p, |1,0\rangle_p, |1,1\rangle_p, \ldots\}$$

- $|\psi\rangle_p = \sum_i a_i |i\rangle_p,\ \sum_i \|a_i\|^2 = 1$

Coin space, $\mathcal{H}_c$

- determines the direction of walker movement
- example basis:

$$\{|\leftarrow\rangle_c, |\rightarrow\rangle_c\}$$
$$\{|\leftarrow\rangle_c, |\rightarrow\rangle_c, |\uparrow\rangle_c, |\downarrow\rangle_c\}$$

- $|\psi\rangle_c = \sum_j b_j |j\rangle_c,\ \sum_j \|b_j\|^2 = 1$

$$|\psi\rangle = |\psi\rangle_p \otimes |\psi\rangle_c, \quad |\psi\rangle \in \mathcal{H}_p \otimes \mathcal{H}_c$$

Coin operator, C

- homogeneous discrete time quantum walk

$$C = 1_p \otimes C'$$

- inhomogeneous discrete time quantum walk

$$C = \sum_i (|i\rangle_p \langle i|_p \otimes C_i)$$

Shift operator, S

$$S = \sum_i \sum_j |v(i,j)\rangle_p \langle i|_p \otimes |j\rangle_c \langle j|_c$$

where $v(i,j)$ denotes the state directly reachable from state i via transition j.

Coin operator, C

- homogeneous discrete time quantum walk

$$C = 1_p \otimes C'$$

- inhomogeneous discrete time quantum walk

$$C = \sum_i (|i\rangle_p \langle i|_p \otimes C_i)$$

$$U = SC$$

Given the initial state of the subsystems $|\psi_0\rangle = |\psi_0\rangle_p \otimes |\psi_0\rangle_c$ and evolution operator $U = SC$, a single step of the discrete time quantum walk can be described by the equation:

$$|\psi_t\rangle = U|\psi_{t-1}\rangle.$$

And the discrete time quantum walk formula has following form:

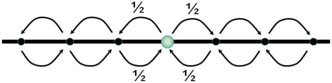
$$|\psi_t\rangle = U^t|\psi_0\rangle.$$

- there is no coin used, only the position space is required,
- transitions can occur at any time according to a time-independent constant γ, describing the jumping rate,
- is used to construct a Hamiltonian matrix H, which in turn is a generator of the evolution operator $U(t)$:

$$U(t) = e^{-iHt}$$

- finally, the continuous time quantum walk is described by the equation:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle.$$



Position space, $\mathcal{H}_p$

- basis:

$$\{\ldots, |-2\rangle_p, |-1\rangle_p, |0\rangle_p, |1\rangle_p, |2\rangle_p, \ldots\}$$

- initial state: $|\psi_0\rangle_p = |0\rangle_p$
- operator:

$$S = \sum_i (|i-1\rangle_p\langle i|_p \otimes |{\leftarrow}\rangle_c\langle{\leftarrow}|_c + \\ +|i+1\rangle_p\langle i|_p \otimes |{\rightarrow}\rangle_c\langle{\rightarrow}|_c)$$

Coin space, $\mathcal{H}_c$

- basis:

$$\{|{\leftarrow}\rangle_c, |{\rightarrow}\rangle_c\}$$

- initial state: $|\psi_0\rangle_c = |{\leftarrow}\rangle_c$
- operator:

$$C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|{\leftarrow}\rangle_c = \frac{|{\leftarrow}\rangle_c + |{\rightarrow}\rangle_c}{\sqrt{2}}$$

$$H|{\rightarrow}\rangle_c = \frac{|{\leftarrow}\rangle_c - |{\rightarrow}\rangle_c}{\sqrt{2}}$$

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Quantum walks

Example - discrete time quantum walk on the infinite line

$$|\psi_0\rangle = |0\rangle_p \otimes |\leftarrow\rangle_c, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Time step	Walker state	Probabilities of finding walker at position										
		-5	-4	-3	-2	-1	0	1	2	3	4	5
0	$\|0\rangle_p \otimes \|\leftarrow\rangle_c$						1					
1	$\frac{1}{\sqrt{2}}(\|-1\rangle_p \otimes \|\leftarrow\rangle_c + \|1\rangle_p \otimes \|\rightarrow\rangle_c)$					$\frac{1}{2}$		$\frac{1}{2}$				
2	$\frac{1}{2}(\|-2\rangle_p \otimes \|\leftarrow\rangle_c + \|0\rangle_p \otimes \|\rightarrow\rangle_c +$ $+\|0\rangle_p \otimes \|\leftarrow\rangle_c) - \|2\rangle_p \otimes \|\rightarrow\rangle_c)$				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3	$\frac{1}{2\sqrt{2}}(\|-3\rangle_p \otimes \|\leftarrow\rangle_c + \|-1\rangle_p \otimes \|\rightarrow\rangle_c +$ $+2\|-1\rangle_p \otimes \|\leftarrow\rangle_c - \|1\rangle_p \otimes \|\leftarrow\rangle_c + \|3\rangle_p \otimes \|\rightarrow\rangle_c)$			$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		
4	$\frac{1}{4}(\|-4\rangle_p \otimes \|\leftarrow\rangle_c + \|-2\rangle_p \otimes \|\rightarrow\rangle_c +$ $+3\|-2\rangle_p \otimes \|\leftarrow\rangle_c + \|0\rangle_p \otimes \|\rightarrow\rangle_c - \|0\rangle_p \otimes \|\leftarrow\rangle_c -$ $-\|2\rangle_p \otimes \|\rightarrow\rangle_c + \|2\rangle_p \otimes \|\leftarrow\rangle_c - \|4\rangle_p \otimes \|\rightarrow\rangle_c)$		$\frac{1}{16}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{1}{16}$	
5	$\frac{1}{4\sqrt{2}}(\|-5\rangle_p \otimes \|\leftarrow\rangle_c + \|-3\rangle_p \otimes \|\rightarrow\rangle_c +$ $+4\|-3\rangle_p \otimes \|\leftarrow\rangle_c + 2\|-1\rangle_p \otimes \|\rightarrow\rangle_c - 2\|1\rangle_p \otimes \|\rightarrow\rangle_c +$ $+2\|3\rangle_p \otimes \|\rightarrow\rangle_c - \|3\rangle_p \otimes \|\leftarrow\rangle_c + \|5\rangle_p \otimes \|\rightarrow\rangle_c)$	$\frac{1}{32}$		$\frac{17}{32}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{32}$		$\frac{1}{32}$

$$|\psi_0\rangle = |0\rangle_p \otimes |{\leftarrow}\rangle_c, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

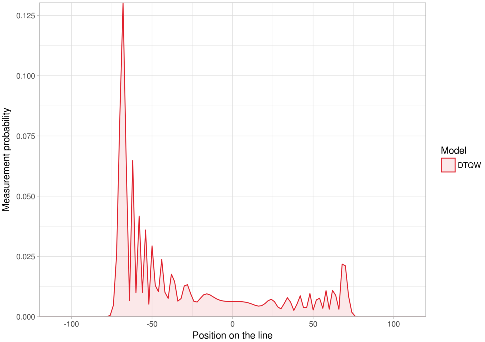

Figure: Probability distribution of measuring walker at given position after 100 steps of discrete time quantum walk with initial state $|\psi_0\rangle = |0\rangle_p \otimes |{\leftarrow}\rangle_c$

Quantum walks

Example - discrete time quantum walk on the infinite line

$$|\psi_0\rangle = |0\rangle_p \otimes |\rightarrow\rangle_c, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Time step	Walker state	Probabilities of finding walker at position										
		-5	-4	-3	-2	-1	0	1	2	3	4	5
0	$\lvert 0\rangle_p \otimes \lvert\rightarrow\rangle_c$						1					
1	$\frac{1}{\sqrt{2}}(\lvert -1\rangle_p \otimes \lvert\leftarrow\rangle_c - \lvert 1\rangle_p \otimes \lvert\rightarrow\rangle_c)$					$\frac{1}{2}$		$\frac{1}{2}$				
2	$\frac{1}{2}(\lvert -2\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert 0\rangle_p \otimes \lvert\rightarrow\rangle_c -$ $- \lvert 0\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert 2\rangle_p \otimes \lvert\rightarrow\rangle_c)$				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3	$\frac{1}{2\sqrt{2}}(\lvert -3\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert -1\rangle_p \otimes \lvert\rightarrow\rangle_c -$ $-2\lvert 1\rangle_p \otimes \lvert\rightarrow\rangle_c + \lvert 1\rangle_p \otimes \lvert\leftarrow\rangle_c - \lvert 3\rangle_p \otimes \lvert\rightarrow\rangle_c)$			$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{8}$		
4	$\frac{1}{4}(\lvert -4\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert -2\rangle_p \otimes \lvert\rightarrow\rangle_c +$ $+ \lvert -2\rangle_p \otimes \lvert\leftarrow\rangle_c - \lvert 0\rangle_p \otimes \lvert\rightarrow\rangle_c - \lvert 0\rangle_p \otimes \lvert\leftarrow\rangle_c +$ $+3\lvert 2\rangle_p \otimes \lvert\rightarrow\rangle_c - \lvert 2\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert 4\rangle_p \otimes \lvert\rightarrow\rangle_c)$		$\frac{1}{16}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{16}$	
5	$\frac{1}{4\sqrt{2}}(\lvert -5\rangle_p \otimes \lvert\leftarrow\rangle_c + \lvert -3\rangle_p \otimes \lvert\rightarrow\rangle_c +$ $+2\lvert -3\rangle_p \otimes \lvert\leftarrow\rangle_c - 2\lvert -1\rangle_p \otimes \lvert\leftarrow\rangle_c + 2\lvert 1\rangle_p \otimes \lvert\leftarrow\rangle_c -$ $-4\lvert 3\rangle_p \otimes \lvert\rightarrow\rangle_c + \lvert 3\rangle_p \otimes \lvert\leftarrow\rangle_c - \lvert 5\rangle_p \otimes \lvert\rightarrow\rangle_c)$	$\frac{1}{32}$		$\frac{5}{32}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{17}{32}$		$\frac{1}{32}$

$$|\psi_0\rangle = |0\rangle_p \otimes |\rightarrow\rangle_c, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

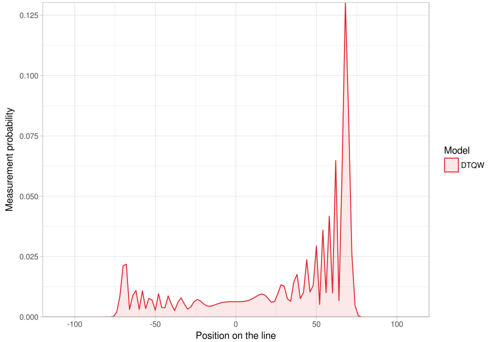

Figure: Probability distribution of measuring walker at given position after 100 steps of discrete time quantum walk with initial state $|\psi_0\rangle = |0\rangle_p \otimes |\rightarrow\rangle_c$

Quantum walks

Example - discrete time quantum walk on the infinite line

$$|\psi_0\rangle = |0\rangle_p \otimes \frac{|{\leftarrow}\rangle_c + |{\rightarrow}\rangle_c}{\sqrt{2}}, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

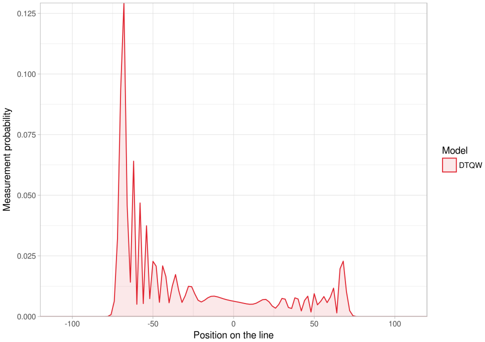

Figure: Probability distribution of measuring walker at given position after 100 steps of discrete time quantum walk with initial state $|\psi_0\rangle = |0\rangle_p \otimes \frac{|{\leftarrow}\rangle_c + |{\rightarrow}\rangle_c}{\sqrt{2}}$

Quantum walks

Example - discrete time quantum walk on the infinite line

$$|\psi_0\rangle = |0\rangle_p \otimes \frac{|{\leftarrow}\rangle_c + i|{\rightarrow}\rangle_c}{\sqrt{2}}, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Time step	Walker state	Probabilities of finding walker at position								
		-4	-3	-2	-1	0	1	2	3	4
0	$\frac{1}{\sqrt{2}}\|0\rangle_p \otimes (\|{\leftarrow}\rangle_c + i\|{\rightarrow}\rangle_c)$					1				
1	$\frac{1}{2}((1+i)\|-1\rangle_p \otimes \|{\leftarrow}\rangle_c + (1-i)\|1\rangle_p \otimes \|{\rightarrow}\rangle_c)$				$\frac{1}{2}$		$\frac{1}{2}$			
2	$\frac{1}{2\sqrt{2}}((1+i)\|-2\rangle_p \otimes \|{\leftarrow}\rangle_c + (1+i)\|0\rangle_p \otimes \|{\rightarrow}\rangle_c +$ $+ (1-i)\|0\rangle_p \otimes \|{\leftarrow}\rangle_c - (1-i)\|2\rangle_p \otimes \|{\rightarrow}\rangle_c)$			$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$		
3	$\frac{1}{4}((1+i)\|-3\rangle_p \otimes \|{\leftarrow}\rangle_c + (1+i)\|-1\rangle_p \otimes \|{\rightarrow}\rangle_c + 2\|-1\rangle_p \otimes \|{\leftarrow}\rangle_c -$ $-2i\|1\rangle_p \otimes \|{\rightarrow}\rangle_c - (1-i)\|1\rangle_p \otimes \|{\leftarrow}\rangle_c + (1-i)\|3\rangle_p \otimes \|{\rightarrow}\rangle_c)$		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$	
4	$\frac{1}{4\sqrt{2}}((1+i)\|-4\rangle_p \otimes \|{\leftarrow}\rangle_c + (1+i)\|-2\rangle_p \otimes \|{\rightarrow}\rangle_c +$ $+(3+i)\|-2\rangle_p \otimes \|{\leftarrow}\rangle_c + (1-i)\|0\rangle_p \otimes \|{\rightarrow}\rangle_c - (1+i)\|0\rangle_p \otimes \|{\leftarrow}\rangle_c -$ $-(1-3i)\|2\rangle_p \otimes \|{\rightarrow}\rangle_c - (1-i)\|2\rangle_p \otimes \|{\leftarrow}\rangle_c + (1-i)\|4\rangle_p \otimes \|{\rightarrow}\rangle_c)$	$\frac{1}{16}$		$\frac{3}{8}$		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{1}{16}$

Quantum walks

Example - discrete time quantum walk on the infinite line

$$|\psi_0\rangle = |0\rangle_p \otimes \frac{|\leftarrow\rangle_c + i|\rightarrow\rangle_c}{\sqrt{2}}, \quad C = 1_p \otimes H = \frac{1}{\sqrt{2}} 1_p \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

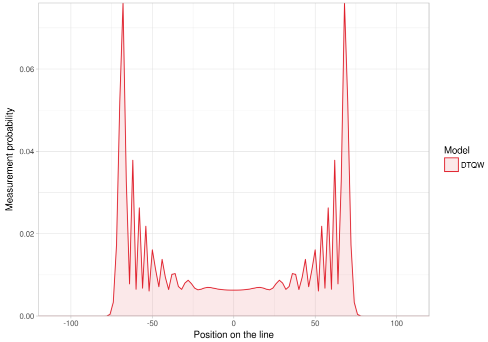

Figure: Probability distribution of measuring walker at given position after 100 steps of discrete time quantum walk with initial state $|\psi_0\rangle = |0\rangle_p \otimes \frac{|\leftarrow\rangle_c + i|\rightarrow\rangle_c}{\sqrt{2}}$

Quantum walks vs classical walks
Comparison of discrete time walks

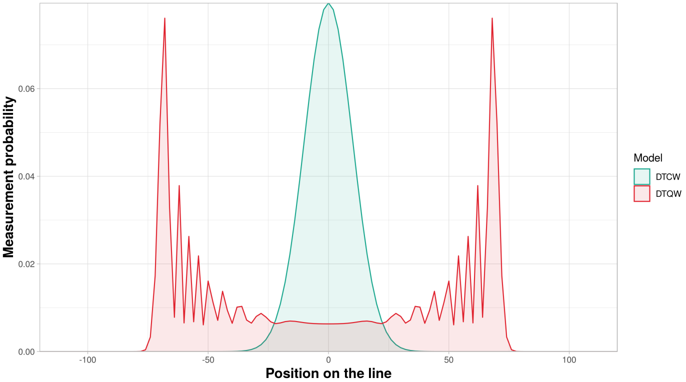

Figure: Probability of finding a walker at given position after 100 steps of: discrete time classical walk (DTCW) and discrete time quantum walk (DTQW)

Quantum walks vs classical walks
Comparison of continuous time walks

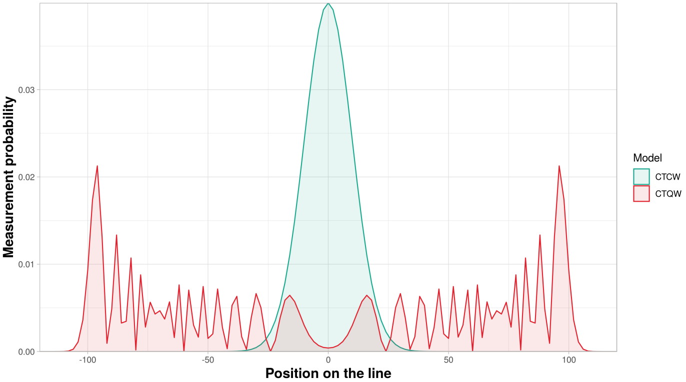

Figure: Probability of finding a walker at given position after time=100 of: continuous time classical walk (CTCW) and continuous time quantum walk (CTQW)

Image segmentation

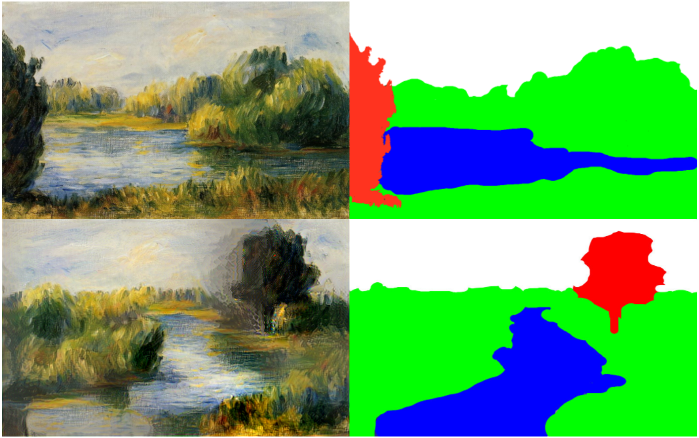

Figure: Example of image segmentation

Definition

Image segmentation is a process of dividing entire image into multiple separate segments – areas inside which pixels expose the similar characteristics (e.g. intensity, hue, texture).

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Definition

Image segmentation can be defined as an activity of assigning each pixel of an image with a label in the way that pixels sharing similar traits of interest are labeled alike.

- Object detection, object recognition
- Traffic monitoring and control
- Video surveillance
- Diagnostics and treatment planing
- Hyper-spectral satellite images analysis

- image – $N \times M$ matrix of pixels p_{ij}
- seeds – set of labeled pixels: (p_{ij}, l_k)
- free parameters:
  - β – parameter related to boundaries amplification
  - γ – rate of spread of continuous time quantum walk
  - T – walk duration

- The input image is transformed into an undirected weighted graph.
- Each pixel is represented by a vertex.
- Each vertex is connected to 4 neighbours forming a regular grid.
- The weight $w_{ij,kl}$ assigned to the edge connecting two pixels p_{ij} and p_{kl} reflects the walker willingness to cross that boundary:

$$w_{ij,kl} = \begin{cases} e^{-\beta\|p_{ij}-p_{kl}\|}, & \text{if } p_{ij} \text{ and } p_{kl} \text{ are connected,} \\ 0, & \text{otherwise.} \end{cases}$$

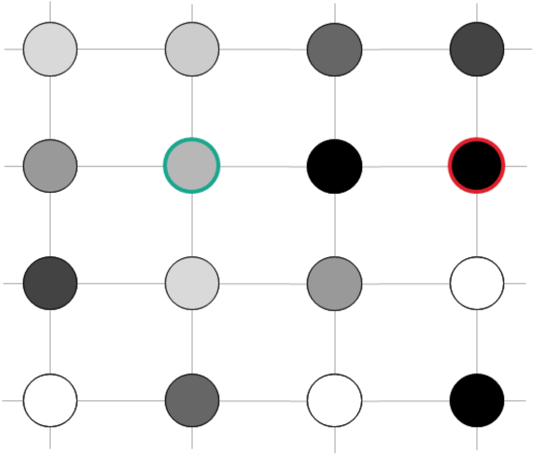

Figure: Input image with seeds

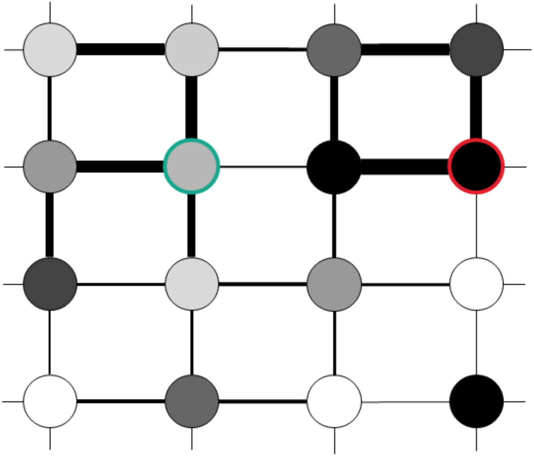

Figure: Input image with weights calculated

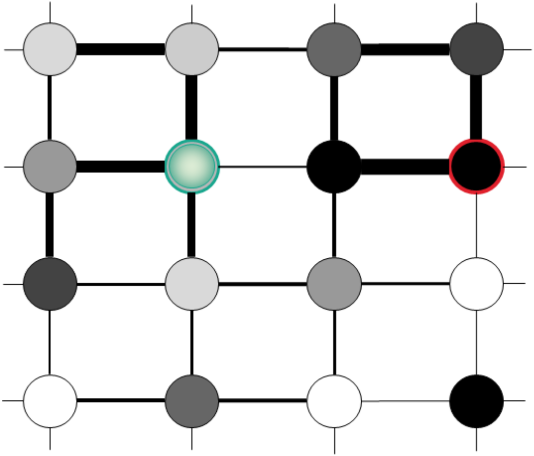

Figure: Quantum walk for first label – initial state

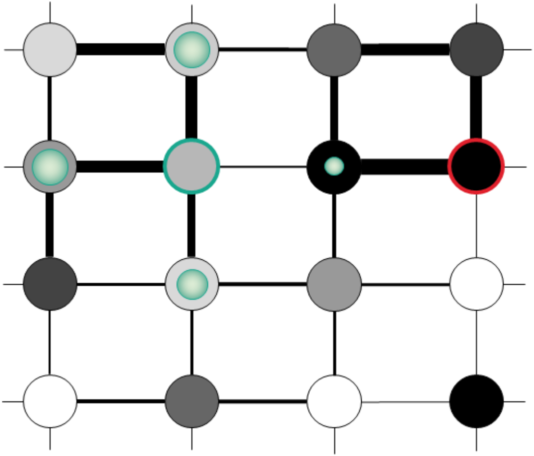

Figure: Quantum walk for first label – after first step

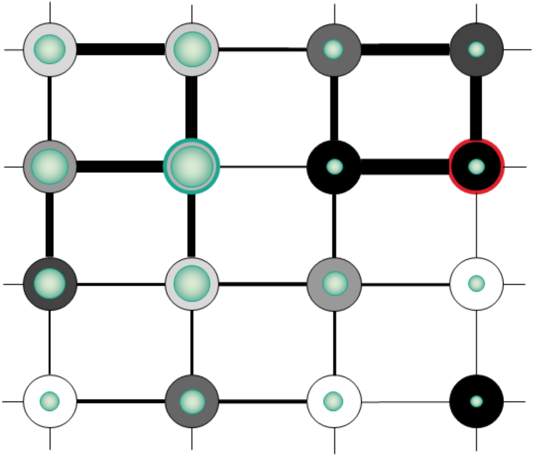

Figure: Quantum walk for first label – final distribution

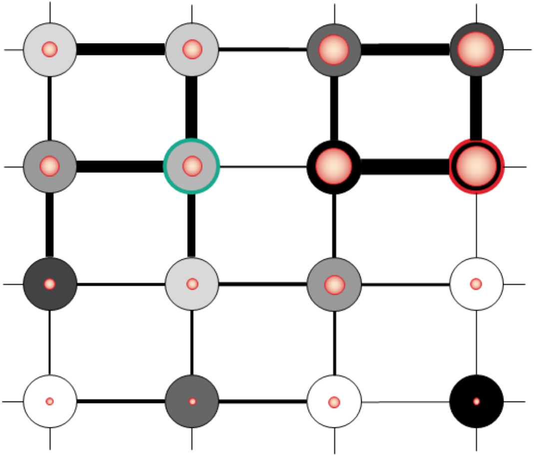

Figure: Quantum walk for second label – final distribution

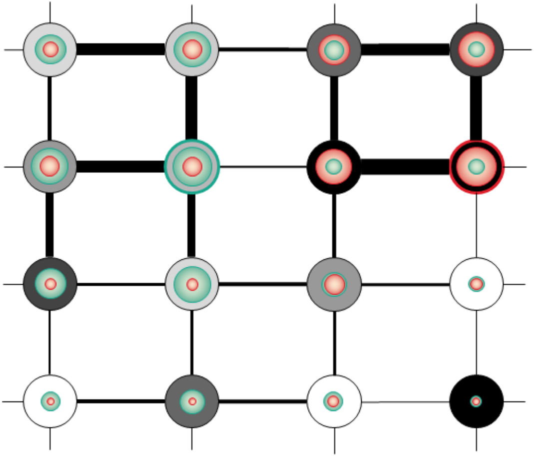

Figure: Juxtaposition of the walk distributions for each label

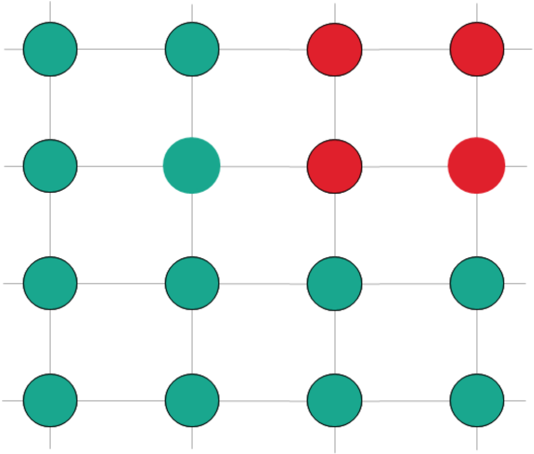

Figure: Image segmentation

Given the input image has $N \times M$ pixels, the position space is spanned by the basis constructed by adding a separate basis state for each pixel:

$$\{|00\rangle_p, |01\rangle_p, ..., |0(M-1)\rangle_p, |10\rangle_p, ..., \\ |(N-1)0\rangle_p, |(N-1)1\rangle_p, ..., |(N-1)(M-1)\rangle_p\}$$

For each label there is performed a separate walk. The initial state of a walk is a normalized superposition of all seeds with given label.

The chosen evolution operator takes the form:

$$U(t) = e^{-iHt},$$

where H is a Hamiltonian constructed based on the weights matrix and a free real parameter γ that determines the rate of spread of the quantum walk:

$$H_{ij,kl} = \begin{cases} \gamma w_{ij,kl}, & \text{if } i \neq k \vee j \neq l. \\ -\gamma \sum_{k',l'} w_{ij,k'l'}, & \text{if } i = k \wedge j = l. \end{cases}$$

Quantum walks do not converge to a stationary distribution. Therefore there was introduced a notion of *limiting distribution*:

Definition

Limiting distribution of the quantum random walk starting from the initial state $|\psi_0\rangle$:

$$\vec{\pi}(|\psi_0\rangle) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \vec{P}_t(|\psi_0\rangle),$$

where $\vec{P}_t(|\psi_0\rangle)$ is probability distribution on position space of measuring the walker starting from initial state $|\psi_0\rangle$ after time t.

The CTQW-LD version of algorithm approximates the limiting distribution of the quantum walk by collecting samples of probability distributions after several time points of the walk and calculating the average distribution.

Despite providing promising results, the CTQW-LD solution has a drawback. It requires obtaining probability distribution for multiple time points and such state tomography is a expensive operation in a quantum system.

CTQW-OS, instead of calculating limiting distribution, utilizes only the state after whole walk, therefore only one state tomography (for each label) is required.

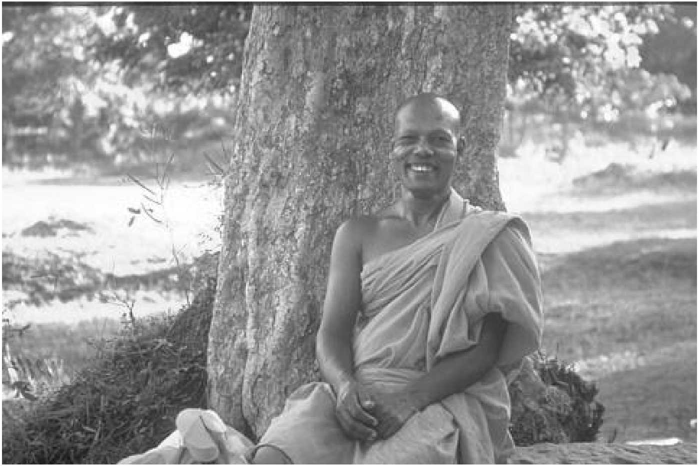

Figure: Input image

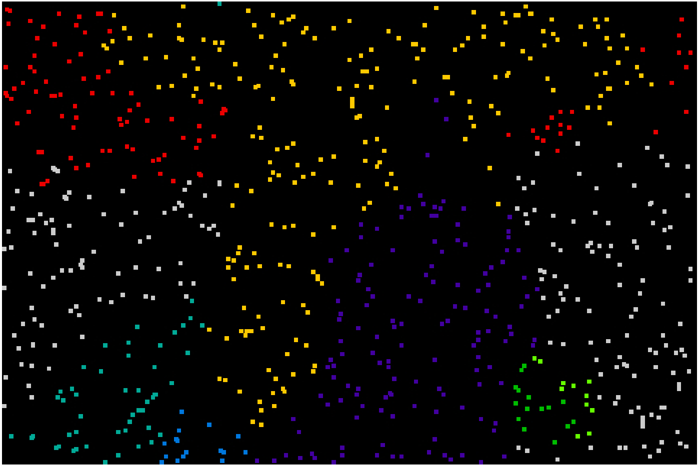

Figure: Seeds

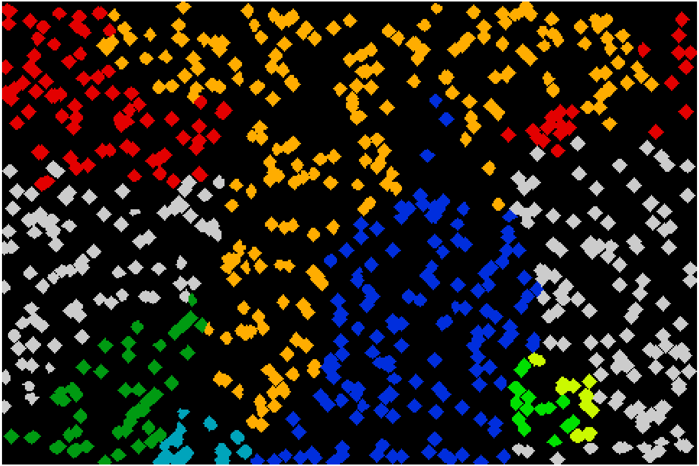

Figure: Walk state after 1 step

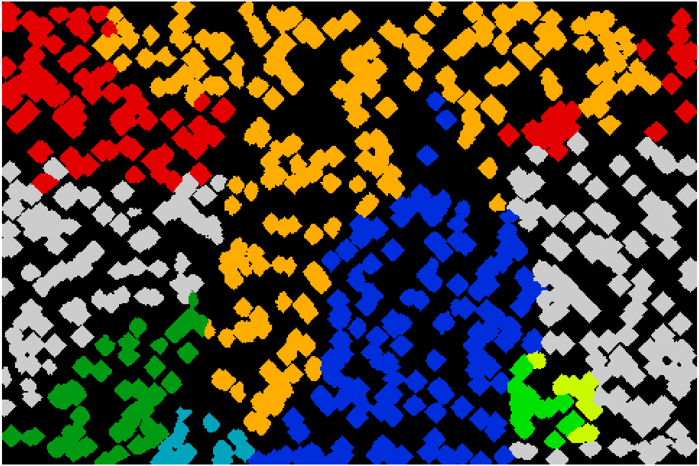

Figure: Walk state after 10 steps

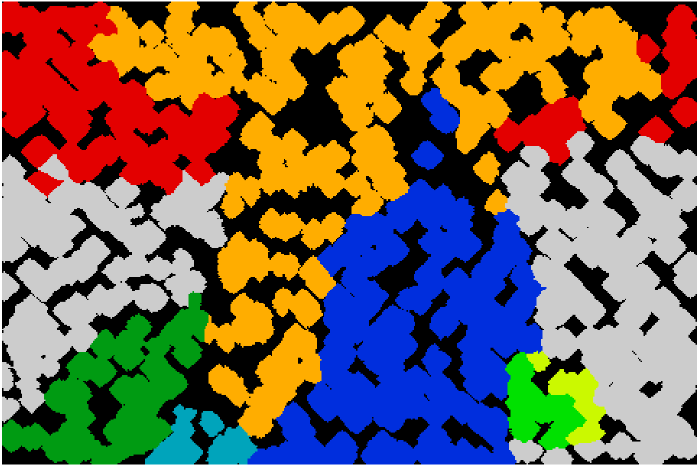

Figure: Walk state after 100 steps

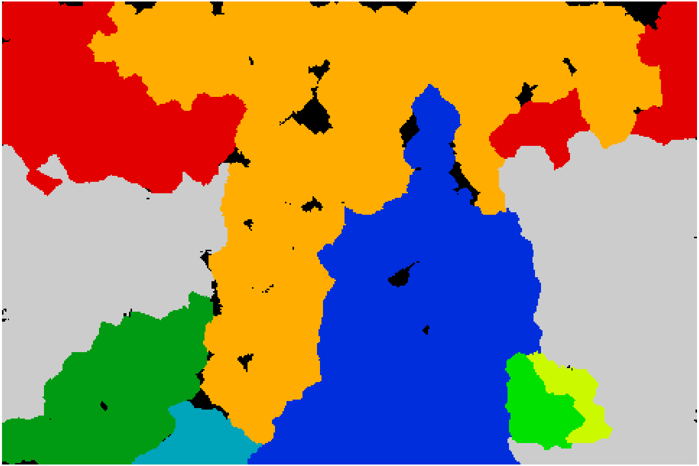

Figure: Walk state after 1000 steps

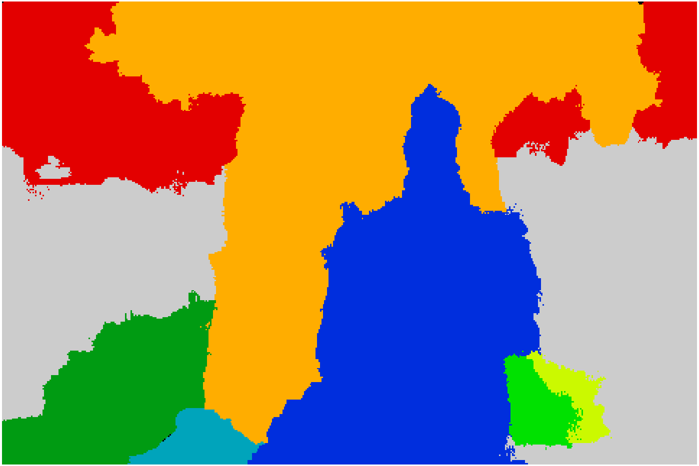

Figure: Walk state after 5000 steps

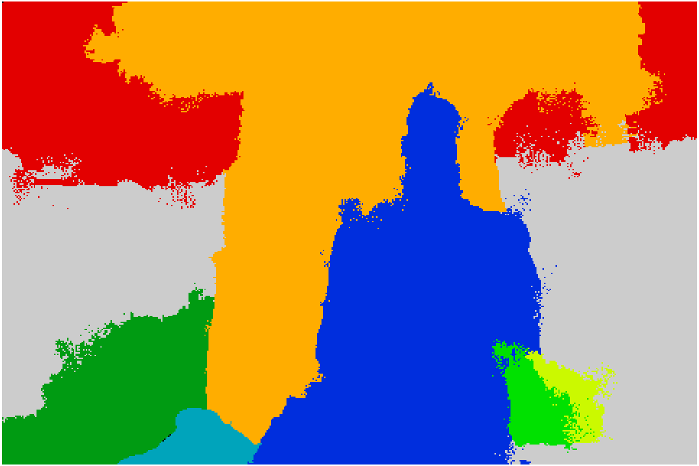

Figure: Walk state after 10000 steps

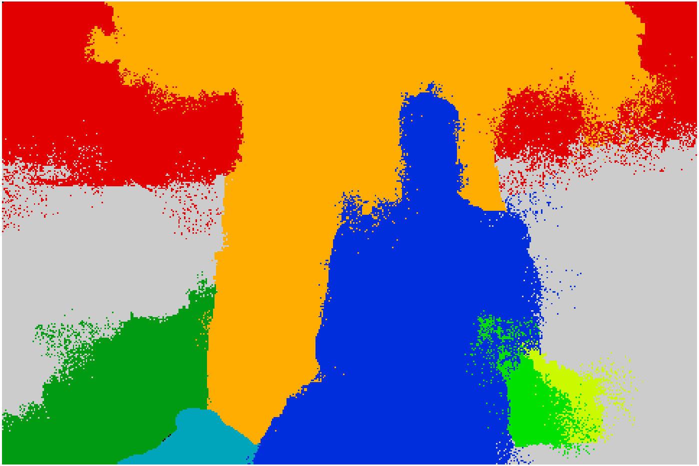

Figure: Walk state after 20000 steps

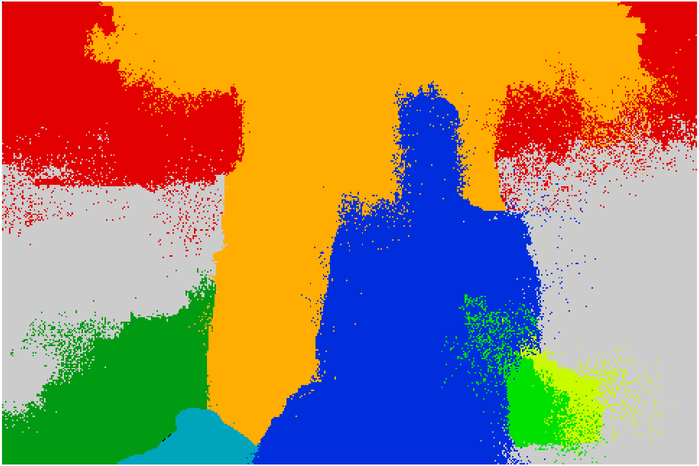

Figure: Walk state after 30000 steps

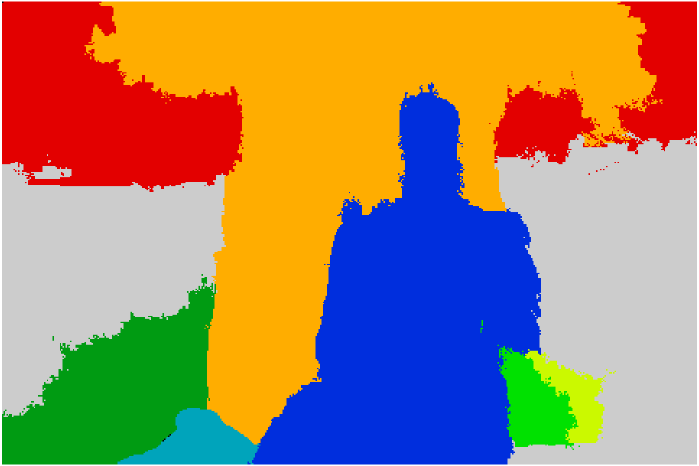

Figure: Image segmentation (CTQW-LD)

Figure: Input image

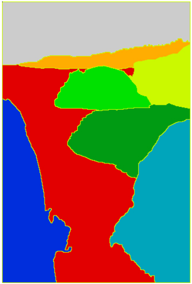

Figure: Ground truth

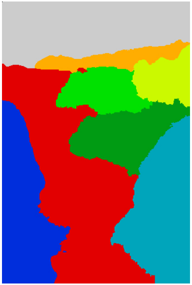

Figure: Grady's segmentation

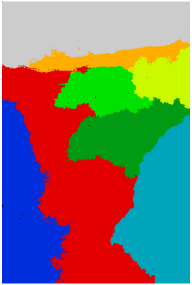

Figure: CTQW-LD

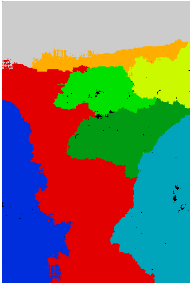

Figure: CTQW-OS

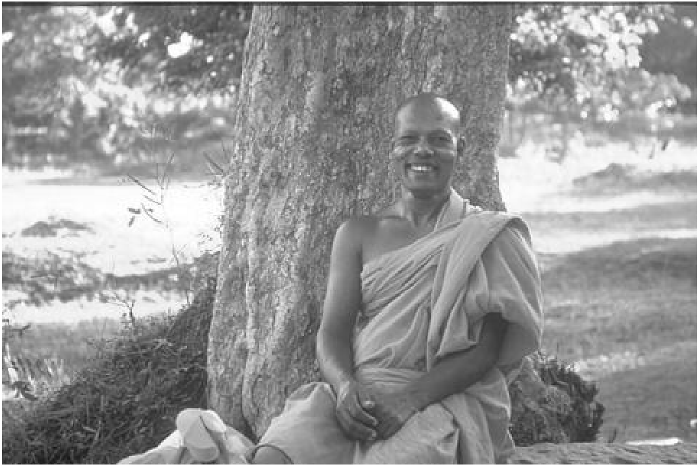

Figure: Input image

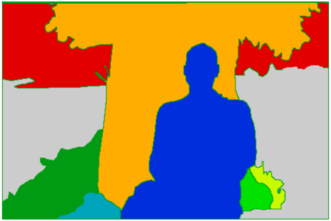

Figure: Ground truth

Figure: Grady's segmentation

Figure: CTQW-LD

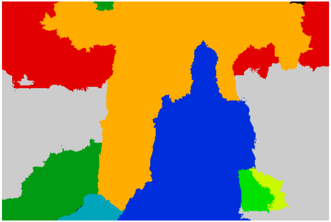

Figure: CTQW-OS

Segmentation method	Manual seeds 0.2%	Random seeds 0.2%	Random seeds 0.5%	Random seeds 1.0%
classical Grady ($\beta = 150.0$)	93.63	90.38	93.50	94.88
classical Grady ($\beta = 90.0$)	93.50	90.25	93.50	94.88
CTQW-LD ($\beta = 150.0, \gamma = 0.001$, $T = 20000$)	93.40	90.89	93.48	94.72
CTQW-LD ($\beta = 90.0, \gamma = 0.0032$, $T = 10000$)	93.33	90.99	93.12	94.22
CTQW-OS ($\beta = 150.0, \gamma = 0.001$, $T = 5000$)	88.60	88.84	92.92	94.26
CTQW-OS ($\beta = 150.0, \gamma = 0.001$, $T = 10000$)	90.86	89.56	92.07	92.92

- Presented quantum walk based segmentation methods give satisfying results that are of similar quality to the method utilizing random walk.
- CTQW-LD solution provides results that are accurate and not too sensitive to the parameters setup.
- CTQW-OS version of algorithm gives a bit less accurate results and is highly sensitive to too long quantum walks. But with the careful choice of parameters the difference is ommitable. It is also much more efficient than CTQW-LD.