

# Quantum game theory and decision optimization

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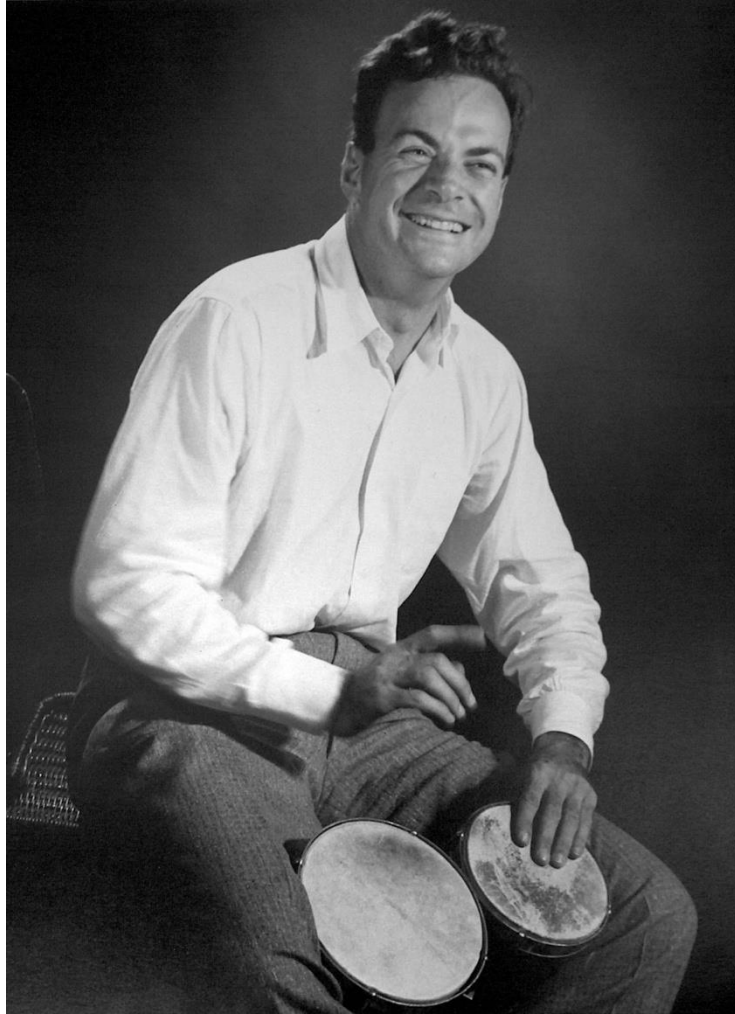
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# About quantum mechanics



Richard Feynman (1918-1988)

*I think it is safe to say that no one understands quantum mechanics. Do not keep saying to yourself, if you can possibly avoid it, "But how can it be like that?" because you will get "down the drain" into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that.*

- Richard Feynman

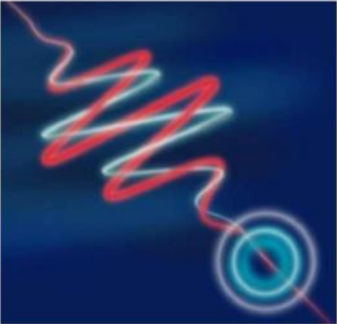
*Those who are not shocked when they first come across quantum mechanics cannot possibly have understood it.*

- Niels Bohr

# Our plan

- Quantum computing
- Equilibria and correlated equilibria in game theory
- EWL approach to quantum game theory
- Pareto efficiency of quantum mixed equilibria
- Quantum absentminded driver
- IBM Q simulations

# Quantum computing advantages



Wave-particle duality → every particle or quantum entity may be described as either a particle or a wave..

- It expresses the inability of the classical concepts "particle" or "wave" to fully describe the behavior of quantum-scale objects.
- **IT IS USED TO INTERACT WITH QUBITS THROUGH INTERFERENCES.**



Probabilistic system → any given state can be observed.

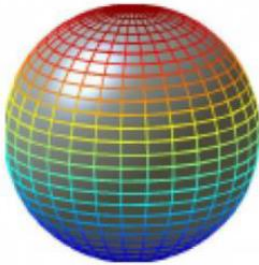
- There is a computable probability corresponding to the likelihood that any given state will be observed if the system is measured.
- Quantum computation is performed by increasing the probability of observing the correct state to a sufficiently high value so that the correct answer may be found with a reasonable amount of certainty.
- **A QUANTUM RESULT IS GENERALLY AN EVALUATION OF THE QUBITS FINAL STATES.**

Bit  
0



1

Qubit  
0



1

In general, a quantum computer with  $n$  qubits can be in any superposition (as Schrodinger's cat) of up to  $2^n$  different states. This compares to a normal computer that can only be in one of these  $2^n$  states at any one time.

# Quantum computing advantages

Superposed states → can be in all possible states at the same time.

$$\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

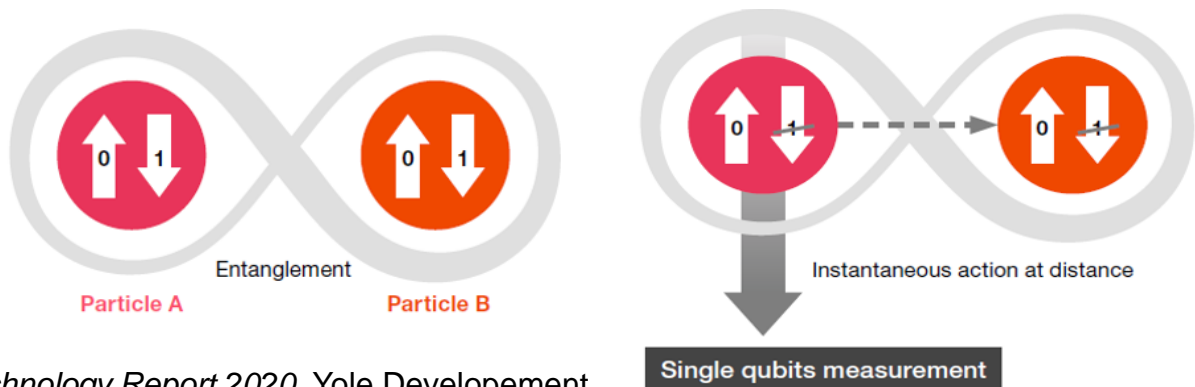


- With respect to a quantum computer, this means that a quantum register exists in a superposition of all its possible configurations of 0's and 1's at the same time, unlike a classical system whose register contains only one value at any given time. It is not until the system is observed that it collapses into an observable, definite classical state. For example, the electron spin can be up and down at the same time.
- **THIS ALLOWS SUPERPOSED CALCULATIONS, THUS DRAMATICALLY DECREASING COMPUTING TIME**

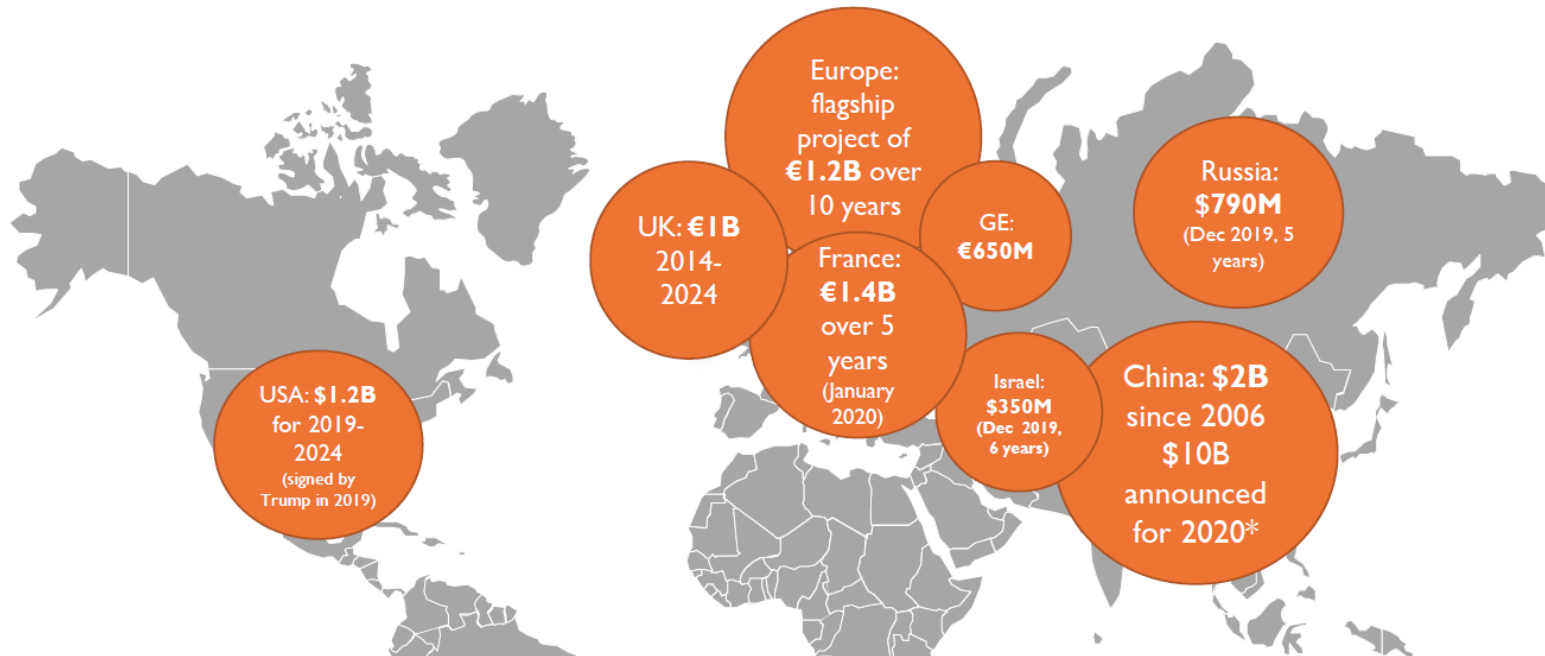
Entanglement → cannot be decomposed into more fundamental part.

- Two distinct elements of a system are entangled if one part cannot be described without taking the other part into consideration.
- An especially interesting quality of quantum entanglement is that elements of a quantum system may be entangled even when they are separated by considerable space.
- Quantum teleportation, an important concept in the field of quantum cryptography, relies on entangled quantum states to send quantum information adequately accurately and over relatively long distances.
- **ENTANGLEMENT IS USED TO LINK THE QUBITS (2 or 3-qubits logic gate) IN QUANTUM COMPUTING AND SYNCHRONIZE THEM.**

This two properties confer a sort of parallelism to a QC and bring the freedom to program designers, to do better than classical equivalents.



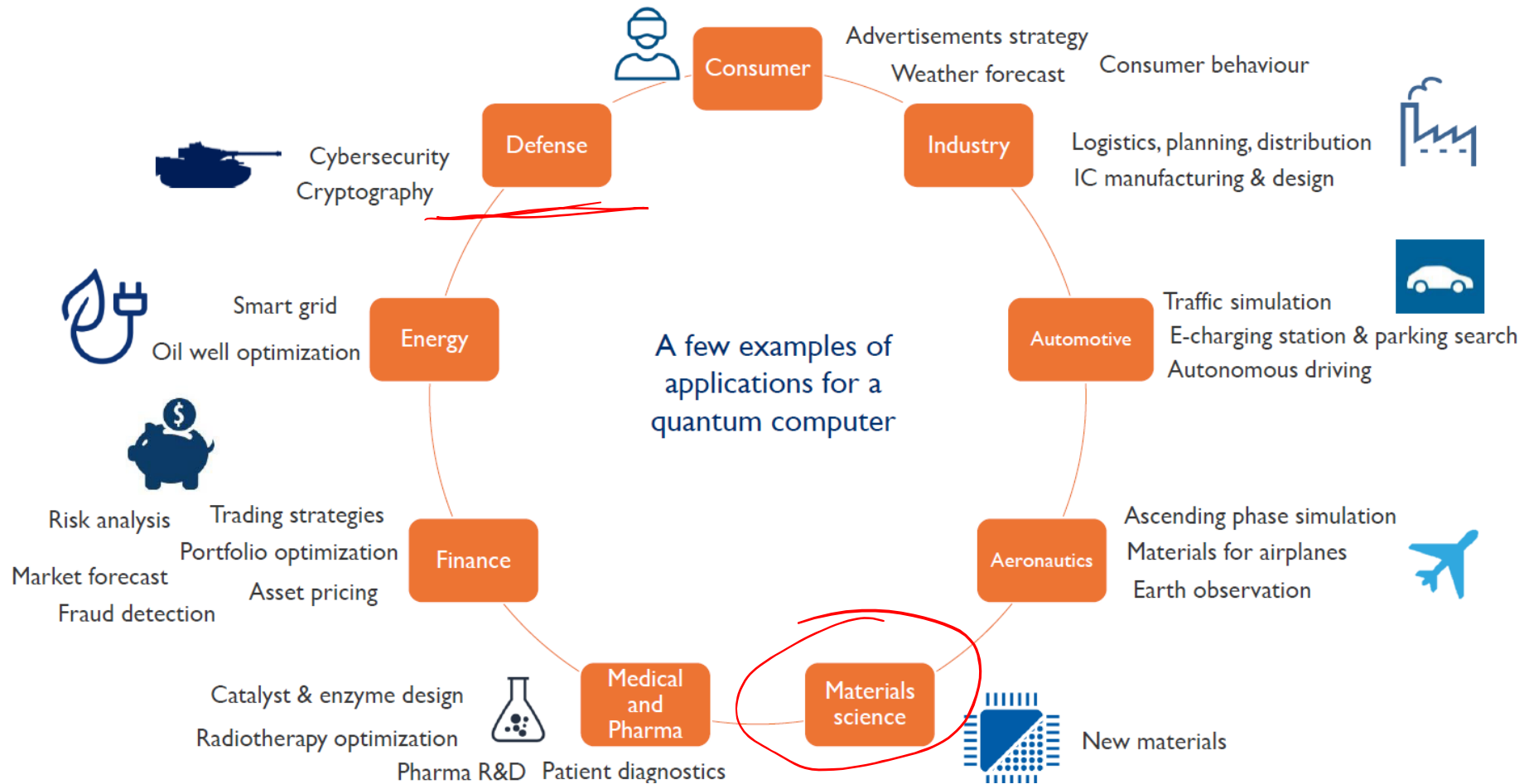
# Quantum public investments



Europe is setting up a quantum effort to compete with US. IN Europe, UK was first to invest in QC (2013). China is also involved in QC (Huawei, Alibaba ...). They are at 10-20 qubits development today, so late compared to US but China wants to be world quantum leader in 2024 (\$10B investment). Map above shows major investments. There are also investment plans in Canada, Australia, Netherlands, Japan, Austria, Singapore.

More than \$16B worldwide

# Quantum computing applications





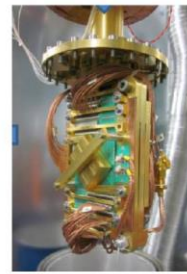
# Physical qubits roadmap



IBM

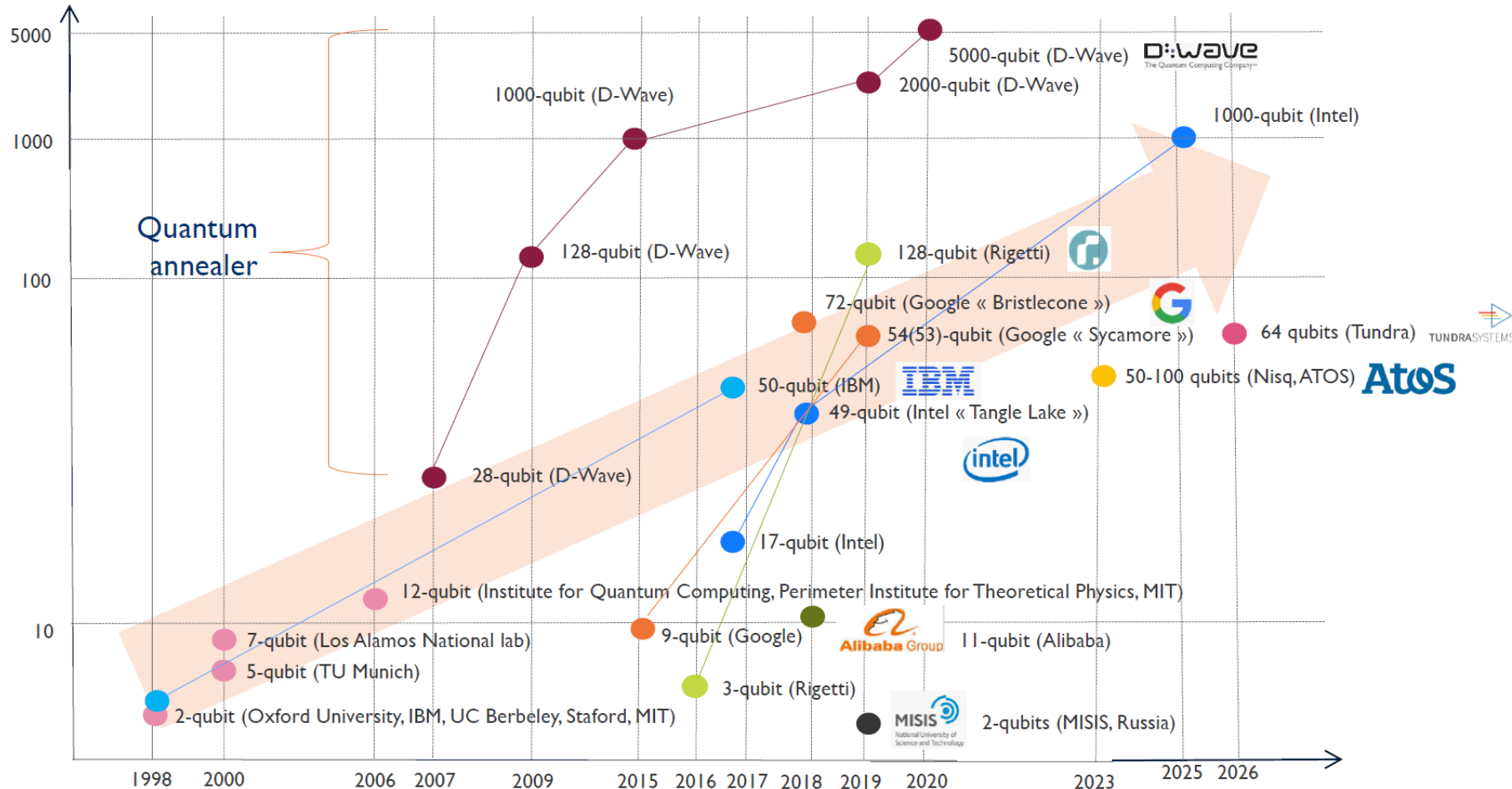


rigetti



D:WAVE

Graph below shows physical qubits roadmap (to be remembered: for a quantum computer, 50 logic qubits minimum are required → it means 5000 physical qubits)

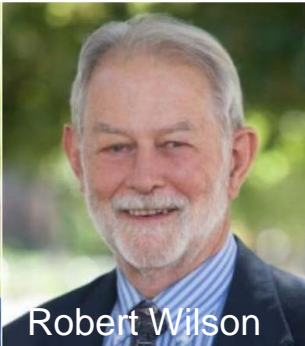




# Game theory and economics



Paul Milgrom



Robert Wilson

„Theory of Games and Economic Behaviour”  
J. von Neumann and O. Morgenstern 1944



Oskar Morgenstern  
i John von Neumann

Nobel prizes for applications of GT to economics:

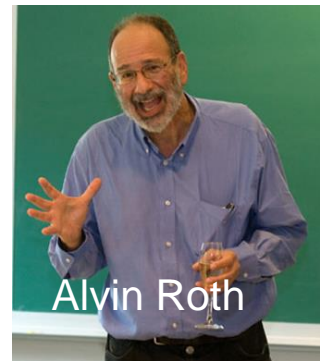
1994 Nash, Harsanyi, Selten *“for their pioneering analysis of equilibria in the theory of non-cooperative games”*

2005 Aumann i Schelling *“for having enhanced our understanding of conflict and cooperation through game-theory analysis”*

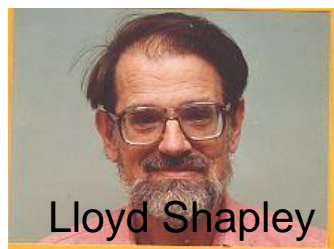
2007 Leonid Hurwicz, Eric Maskin i Roger Myerson *“for having laid the foundations of mechanism design theory”*

2012 Alvin Roth, Lloyd Shapley *“for the theory of stable allocations and the practice of market design”*.

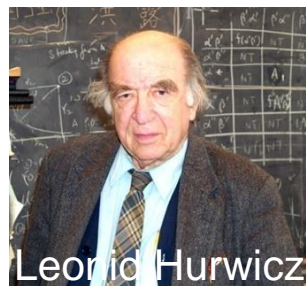
2020 Raul Milgrom, Robert Wilson *“for improvements to auction theory and inventions of new auction formats.”*



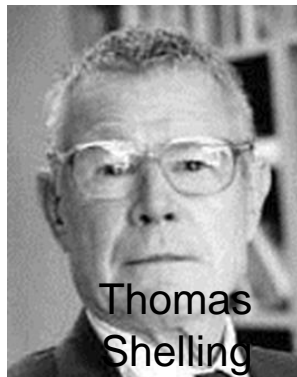
Alvin Roth



Lloyd Shapley



Leonid Hurwicz



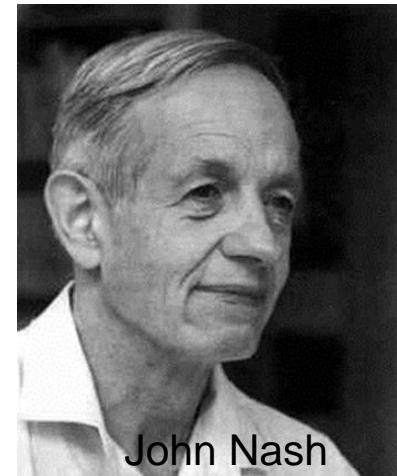
Thomas  
Schelling



Robert Aumann



Reinhard  
Selten



John Nash



2001 “Beatiful mind”  
movie about Nash

# Games and probability distributions

We consider two player *games*

$$G = \left( N, \{S_X\}_{X \in N}, \{P_X\}_{X \in N} \right)$$

where:

$N = \{A, B\}$  is the set of players

$S_A = \{A_0, A_1\}$ ,  $S_B = \{B_0, B_1\}$  are possible pure strategies

$P_X: S_A \times S_B \rightarrow \{v_{ij}^X \in \mathbb{R} \mid i, j = 0, 1\}$ ,  $X = A, B$ , are payoff functions, represented by the game bimatrix

$$\begin{pmatrix} (v_{00}^A, v_{00}^B) & (v_{01}^A, v_{01}^B) \\ (v_{10}^A, v_{10}^B) & (v_{11}^A, v_{11}^B) \end{pmatrix}$$

Let

$$\Delta(S_A \times S_B) = \left\{ \sum_{i,j=0,1} \sigma_{ij} A_i B_j \mid \sigma_{ij} \geq 0, \sum_{i,j=0,1} \sigma_{ij} = 1 \right\}$$

be the set of *probability distributions* over  $S_A \times S_B$

# Mixed strategies and Nash equilibria

If the set of probability distributions can be factorized

$$p + q = 1$$

$$\begin{matrix} p \\ q \end{matrix} \begin{matrix} \sigma_A \\ (1 - \sigma_A) \end{matrix}$$

$$\begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} = \begin{pmatrix} \sigma_A \sigma_B & \sigma_A (1 - \sigma_B) \\ (1 - \sigma_A) \sigma_B & (1 - \sigma_A) (1 - \sigma_B) \end{pmatrix}$$

they define *mixed strategies*  $\sigma_A, \sigma_B \in [0, 1]$ .

$$1 - \sigma_B$$

The *mixed classical game* is

$$G^{mix} = (N, \Delta S_A, \Delta S_B, \Delta P_A, \Delta P_B)$$

where  $\Delta(S_X) = \{\sigma_X X_0 + (1 - \sigma_X) X_1 \mid 0 \leq \sigma_X \leq 1\} \equiv [0, 1]$ .

Mixed strategies form a subset of all probability distributions

$$\Delta S_A \times \Delta S_B \subset \Delta(S_A \times S_B)$$

The pair of strategies  $(\sigma_A^*, \sigma_B^*) \in \Delta S_A \times \Delta S_B$  is a Nash equilibrium

iff  $\Delta P_A(\sigma_A^*, \sigma_B^*) \geq \Delta P_A(\sigma_A, \sigma_B^*)$  and  $\Delta P_B(\sigma_A^*, \sigma_B^*) \geq \Delta P_B(\sigma_A^*, \sigma_B)$ ,

for each  $\sigma_X \in \Delta S_X, X = A, B$



The executor performance depends on the goalkeeper's strategy



# Pareto optimality and correlated equilibria

A pair of strategies  $(\sigma_A, \sigma_B) \in S$  is not Pareto optimal in  $S$  if there exists another pair  $(\sigma_A', \sigma_B') \in S$  that is better for one of the players and not worse for the other. Otherwise  $(\sigma_A, \sigma_B) \in S$  is called *Pareto optimal*.

Probability distribution  $\{\sigma_{ij}\}_{i,j=0,1}$  over set of strategies  $(A_i, B_j)_{i,j=0,1}$  of the game  $G$  is a correlated equilibrium iff

$$\sum_{j=0,1} \sigma_{ij} v_{ij}^A \geq \sum_{j=0,1} \sigma_{ij} v_{-ij}^A \quad \text{and} \quad \sum_{j=0,1} \sigma_{ji} v_{ji}^B \geq \sum_{j=0,1} \sigma_{ji} v_{j(-i)}^B$$

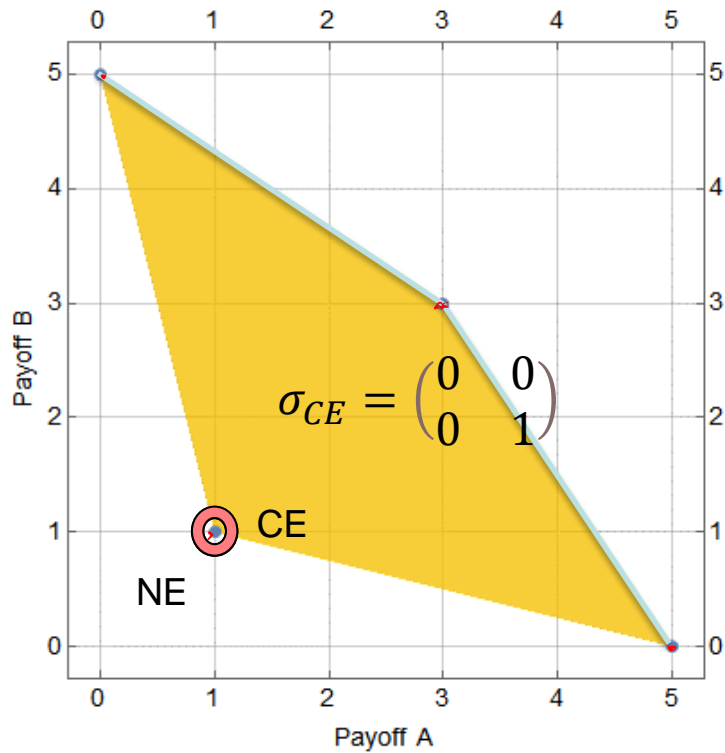
where  $-i \neq i$  is the index of the remaining strategy.



# Efficiency of selected classical games

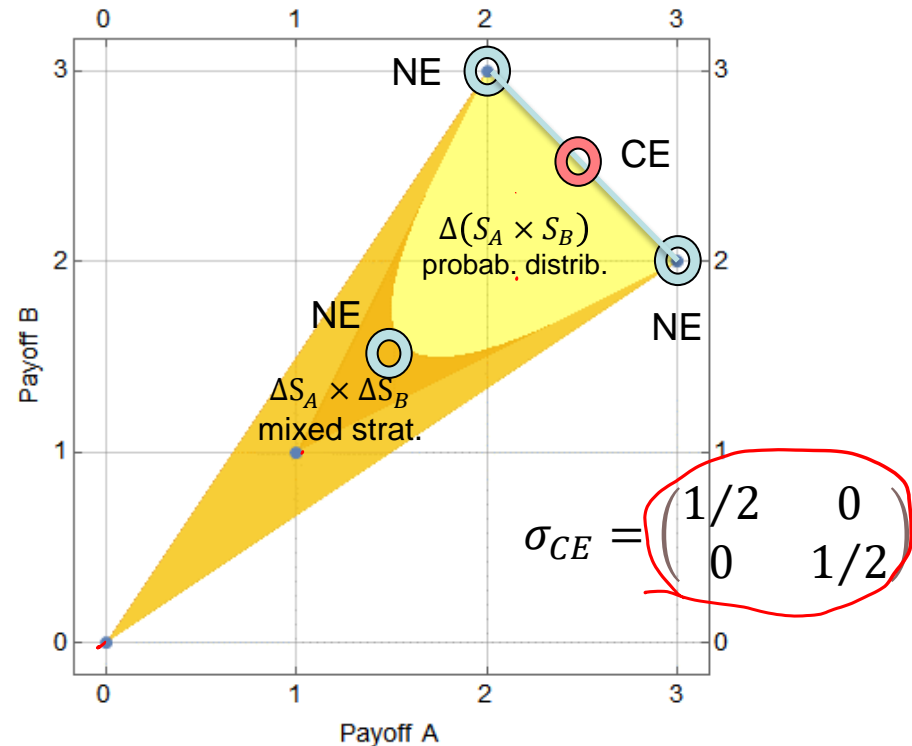
prisoner's dilemma		Bob	
		$B_0$	$B_1$
Alice	$A_0$	(3, 3)	(0, 5)
	$A_1$	(5, 0)	(1, 1)

Correlated equilibrium  $\left\{ \begin{array}{l} \sigma_{00} = \sigma_{01} = \sigma_{10} = 0 \\ \sigma_{11} = 1 \end{array} \right.$



battle of the sexes		Bob	
		$B_0$	$B_1$
Alice	$A_0$	(3, 2)	(1, 1)
	$A_1$	(0, 0)	(2, 3)

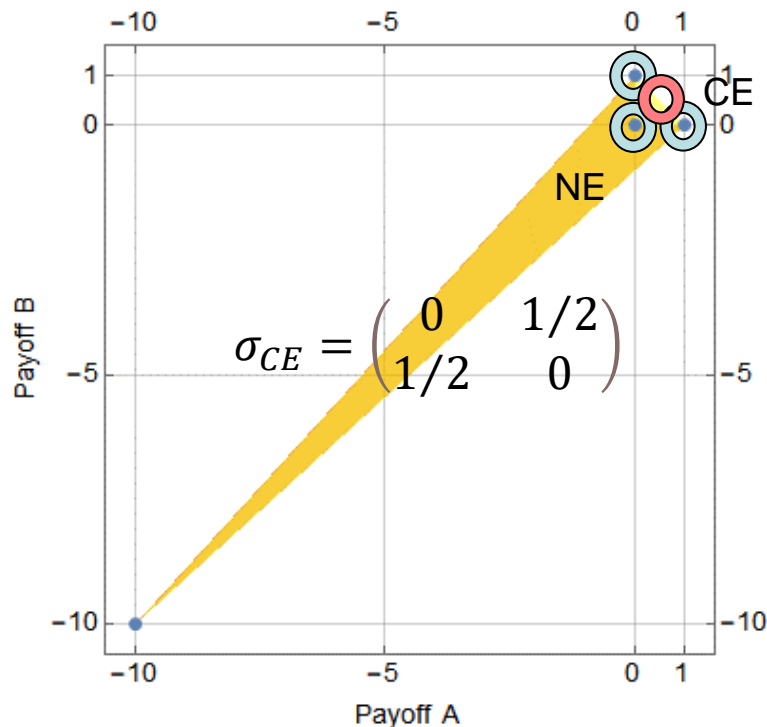
Correlated equilibrium  $\left\{ \begin{array}{l} 3\sigma_{00} \geq \sigma_{01}, \sigma_{00} \geq 3\sigma_{10} \\ 3\sigma_{11} \geq \sigma_{01}, \sigma_{11} \geq 3\sigma_{10} \end{array} \right.$



# Efficiency of selected classical games

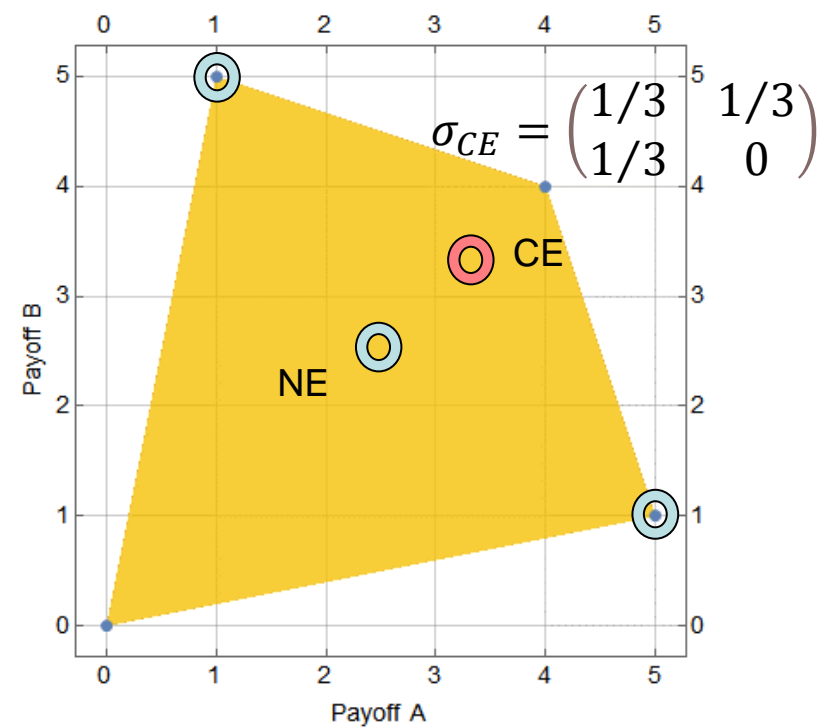
chicken		Driver B	
Driver A		$B_0$	$B_1$
	$A_0$	(0, 0)	(0, 1)
	$A_1$	(1, 0)	(-10, -10)

Correlated equilibrium  $\left\{ \begin{array}{l} \sigma_{00} \leq 10\sigma_{01}, \sigma_{00} \leq 10\sigma_{10} \\ 10\sigma_{11} \leq \sigma_{01}, 10\sigma_{11} \leq \sigma_{10} \end{array} \right.$



chicken 2		Player B	
Player A		$B_0$	$B_1$
	$A_0$	(4, 4)	(1, 5)
	$A_1$	(5, 1)	(0, 0)

Correlated equilibrium  $\left\{ \begin{array}{l} \sigma_{00} \leq \sigma_{01}, \sigma_{00} \leq \sigma_{10} \\ \sigma_{11} \leq \sigma_{01}, \sigma_{11} \leq \sigma_{10} \end{array} \right.$





# Quantum game preliminaries

The standard quantum game in Eisert-Wilkens-Lewenstein quantization scheme is: (Eisert et al, PRL 83, 3077 (1999))

$$\Gamma_{EWL} = (N, \{U_X\}_{X \in N}, \{\Pi_X\}_{X \in N})$$

where:

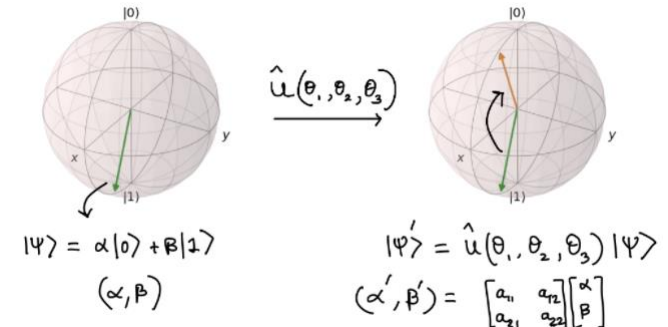
$N = \{A, B\}$  is the set of players

The unitary transformations  $\hat{U}_A = \hat{U}(\theta_A, \alpha_A, \beta_A)$ ,  $\hat{U}_B = \hat{U}(\theta_B, \alpha_B, \beta_B)$

$$\hat{U}(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X} \cos \frac{\theta_X}{2} & ie^{i\beta_X} \sin \frac{\theta_X}{2} \\ ie^{-i\beta_X} \sin \frac{\theta_X}{2} & e^{-i\alpha_X} \cos \frac{\theta_X}{2} \end{pmatrix},$$

$\theta_X \in [0, \pi]$ ,  $\alpha_X, \beta_X \in [0, 2\pi]$ ,  $X = A, B$

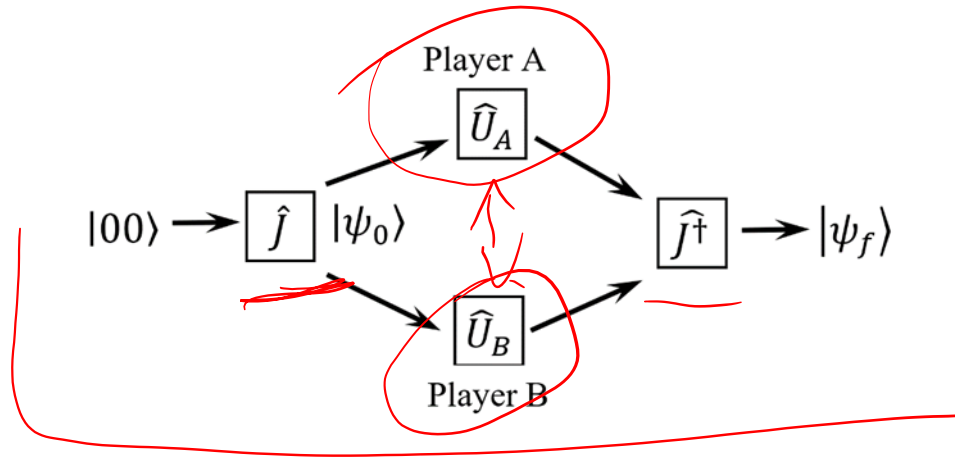
are quantum strategies.





# EWL approach

The quantum EWL approach to the game is



where:  $|00\rangle$  is the initial state

$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I} + i\sigma_x \otimes \sigma_x)$ ,  $J^\dagger$  are the entangling, disentangling operators,

$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X} \cos \frac{\theta_X}{2} & ie^{i\beta_X} \sin \frac{\theta_X}{2} \\ ie^{-i\beta_X} \sin \frac{\theta_X}{2} & e^{-i\alpha_X} \cos \frac{\theta_X}{2} \end{pmatrix}, X = A, B,$$

$|\psi_f\rangle = \sum_{i,j=0,1} p_{ij} |ij\rangle$ , is the final state defining the game payoffs

# Quantum game payoffs

$\Pi_X: SU(2) \times SU(2) \rightarrow \mathbb{R}$  are payoff functions defined by:

$$\Pi_X(\hat{U}_A, \hat{U}_B, \gamma) = \sum_{k,l=0}^1 v_{k,l}^X |\langle \Psi_{k,l}(\gamma) | \hat{U}_A \otimes \hat{U}_B | \Psi(\gamma) \rangle|^2, \quad X = A, B$$

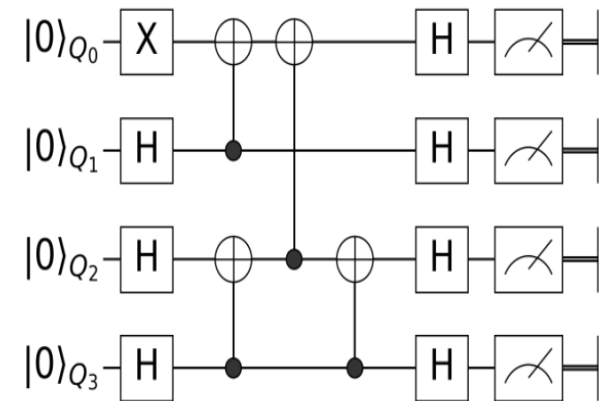
$$|\Psi_{k,l}(\gamma)\rangle = C_k \otimes C_l |\Psi(\gamma)\rangle$$

In case of a fully quantum case  $\gamma = \pi/2$  :

$$\Pi_X(\hat{U}_A, \hat{U}_B) = \sum_{k,l=0,1} |p_{kl}|^2 v_{kl}^X, \quad X = A, B,$$

where:

$$\begin{aligned} |p_{00}|^2 &= \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\alpha_A + \alpha_B) + \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \sin(\beta_A + \beta_B), \\ |p_{01}|^2 &= \cos \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos(\alpha_A - \beta_B) + \sin \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \sin(\alpha_B - \beta_A), \\ |p_{10}|^2 &= \cos \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \sin(\alpha_A - \beta_B) + \sin \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\alpha_B - \beta_A), \\ |p_{11}|^2 &= \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \sin(\alpha_A + \alpha_B) - \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos(\beta_A + \beta_B). \end{aligned}$$



# Quantum Pauli strategies

Strategies  $\hat{U}_A = \hat{U}(\theta_A, \alpha_A, \beta_A)$  and  $\hat{U}_B = \hat{U}(\theta_B, \alpha_B, \beta_B)$ ,

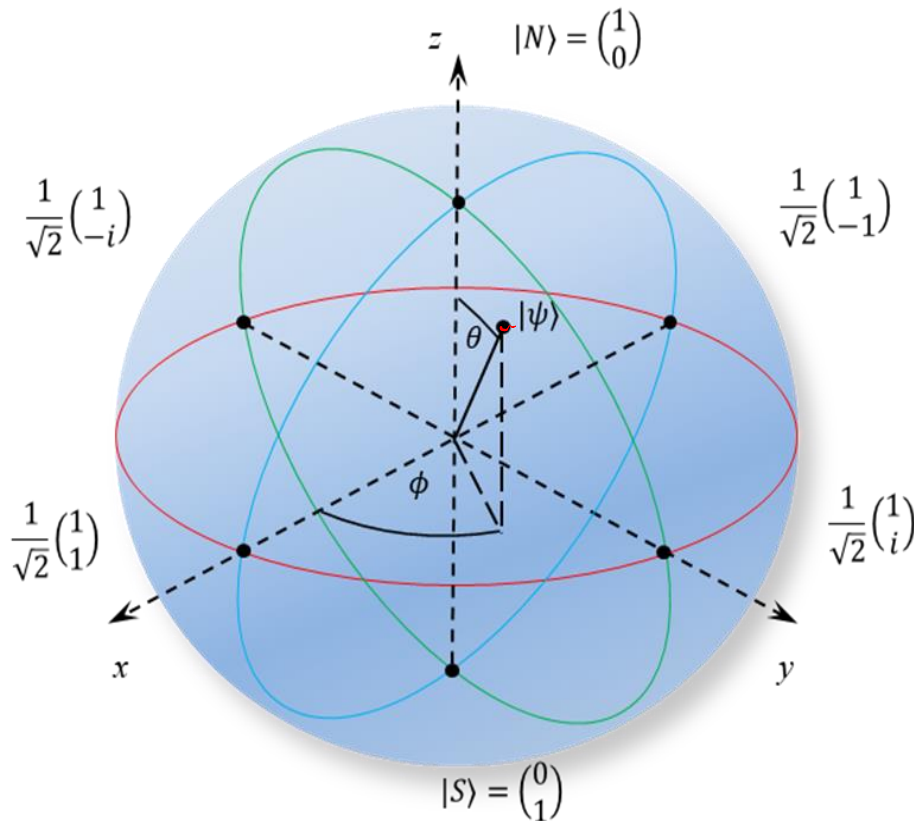
$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X} \cos \frac{\theta_X}{2} & ie^{i\beta_X} \sin \frac{\theta_X}{2} \\ ie^{-i\beta_X} \sin \frac{\theta_X}{2} & e^{-i\alpha_X} \cos \frac{\theta_X}{2} \end{pmatrix}, \text{ are generated by Pauli strategies:}$$

$$\hat{P}_0 = \hat{U}(0, 0, \beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{P}_x = \hat{U}(\pi, \alpha, \pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

$$\hat{P}_y = \hat{U}(\pi, \alpha, \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{P}_z = \hat{U}(0, \pi/2, \beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$



where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

are Pauli matrices

# Classical limit of the quantum game

Let us assume  $\alpha = \beta = 0$ , in this case

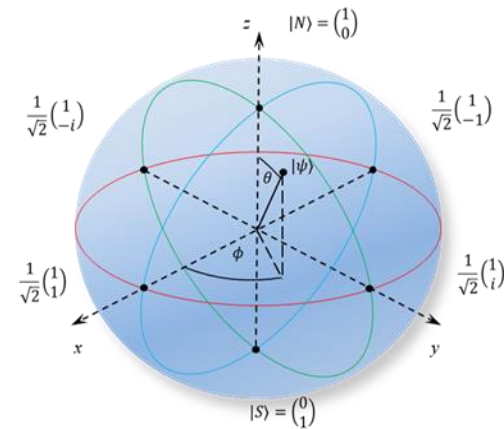
$$\hat{U}(\theta, 0, 0) = \cos \frac{\theta}{2} \hat{I} + i \sin \frac{\theta}{2} \sigma_x$$

is equivalent to the classical mixed strategy

		B	
		$\cos^2 \frac{\theta_B}{2}$	$\sin^2 \frac{\theta_B}{2}$
A	$\cos^2 \frac{\theta_A}{2}$	$(a_{00}, b_{00})$	$(a_{01}, b_{01})$
	$\sin^2 \frac{\theta_A}{2}$	$(a_{10}, b_{10})$	$(a_{11}, b_{11})$
		$\theta_B = 0$	$\theta_B = \pi$

and the payoffs are

$$\begin{aligned} \$_{A(B)} = & a(b)_{00} \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + a(b)_{01} \cos^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\ & + a(b)_{10} \sin^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + a(b)_{11} \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \end{aligned}$$



# EWL with Frąckiewicz-Pykacz parameterization

Let us restrict the set of quantum strategies to

$$\hat{U}_X(\theta_X, \phi_X) = \begin{pmatrix} e^{-i\phi_X} \cos \frac{\theta_X}{2} & -e^{-i\phi_X} \sin \frac{\theta_X}{2} \\ e^{i\phi_X} \sin \frac{\theta_X}{2} & e^{i\phi_X} \cos \frac{\theta_X}{2} \end{pmatrix}$$

$$\left\{ \begin{aligned} \hat{P}_0 &= \hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \hat{P}_x &= \hat{U}\left(\pi, \frac{3\pi}{2}\right) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\ \hat{P}_y &= \hat{U}(\pi, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \hat{P}_z &= \hat{U}\left(0, \frac{3\pi}{2}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned} \right.$$

- In this parameterization, there are Nash equilibria in pure strategies which do not exist for SU(2) parameterization
- F-P parametrization is invariant with respect to strongly isomorphic transformation of input games

# Quantum game in Pauli strategies

The payoff matrix of Pauli strategies in the EWL scheme

		Player B			
		$\widehat{P}_0$	$\widehat{P}_x$	$\widehat{P}_y$	$\widehat{P}_z$
Player A	$\widehat{P}_0$	$(a_{00}, b_{00})$	$(a_{01}, b_{01})$	$(a_{10}, b_{10})$	$(a_{11}, b_{11})$
	$\widehat{P}_x$	$(a_{10}, b_{10})$	$(a_{11}, b_{11})$	$(a_{00}, b_{00})$	$(a_{01}, b_{01})$
	$\widehat{P}_y$	$(a_{01}, b_{01})$	$(a_{00}, b_{00})$	$(a_{11}, b_{11})$	$(a_{10}, b_{10})$
	$\widehat{P}_z$	$(a_{11}, b_{11})$	$(a_{10}, b_{10})$	$(a_{01}, b_{01})$	$(a_{00}, b_{00})$

one can also construct mixed Pauli strategies defined by quadruples of coefficients:

$$\Delta U_X \equiv \Delta(U_X) = \left\{ \sum_{\alpha=0,x,y,z} \sigma_{\alpha}^X \widehat{P}_{\alpha} \mid 0 \leq \sigma_{\alpha}^X; \sum_{\alpha=0,x,y,z} \sigma_{\alpha}^X = 1 \right\}, \quad X = A, B,$$

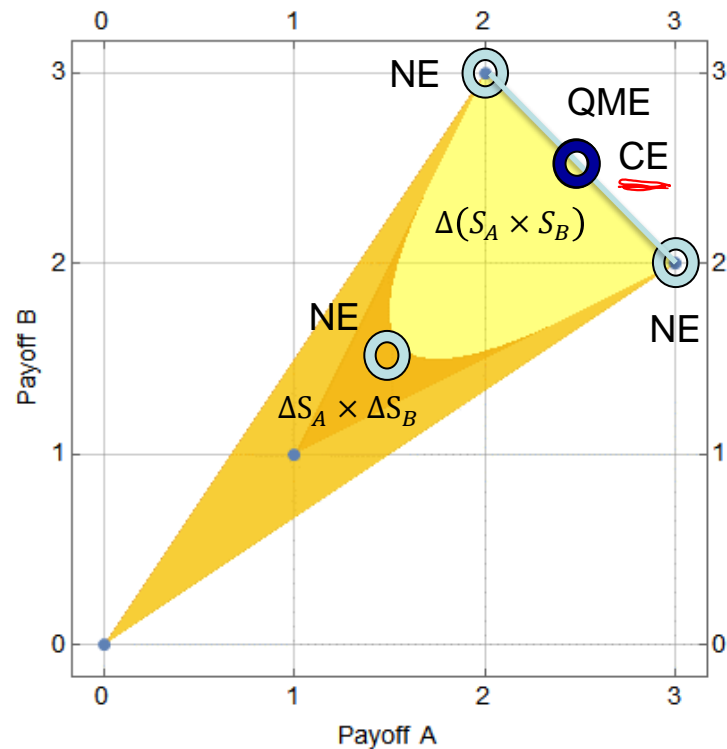
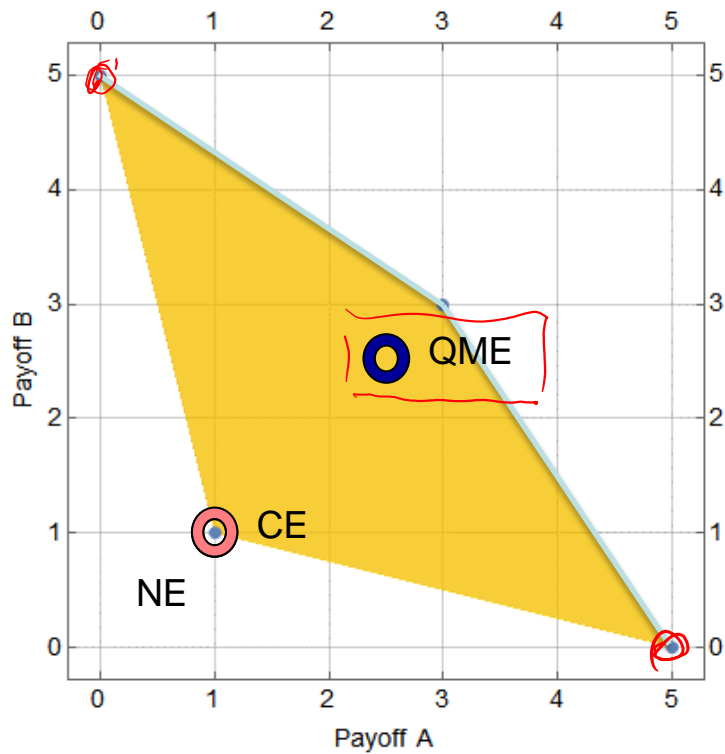
# Quantum Mixed Equilibria

prisoner's dilemma		Bob	
		$B_0$	$B_1$
Alice	$A_0$	(3, 3)	(0, 5)
	$A_1$	(5, 0)	(1, 1)

battle of the sexes		Bob	
		$B_0$	$B_1$
Alice	$A_0$	(3, 2)	(1, 1)
	$A_1$	(0, 0)	(2, 3)

$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\sigma^A = \sigma^B = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

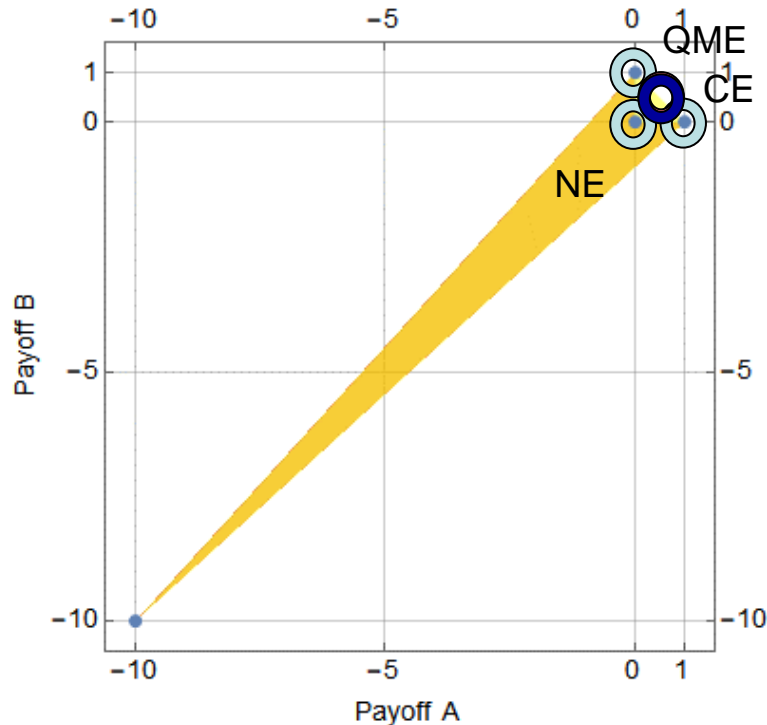




# Quantum Mixed Equilibria

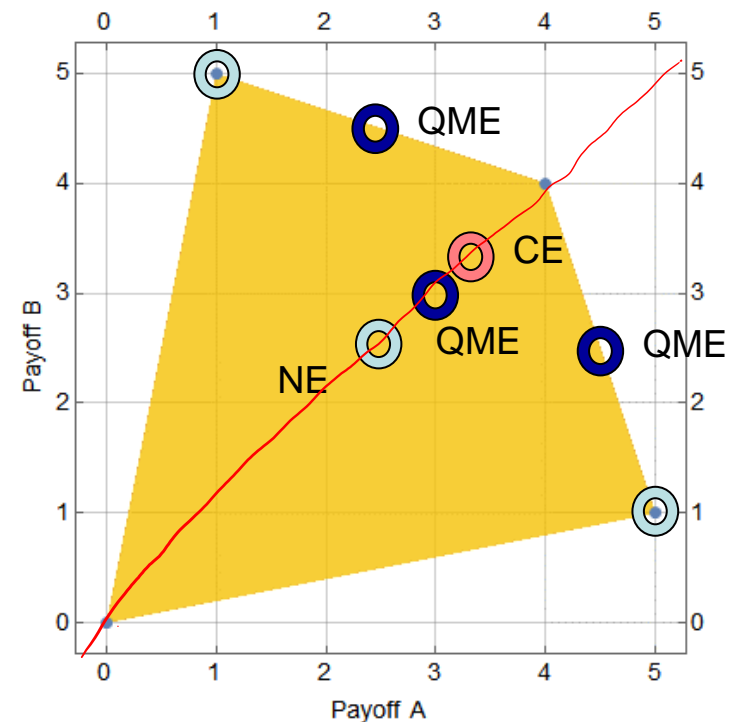
chicken		Driver B	
Driver A		$B_0$	$B_1$
	$A_0$	(0, 0)	(0, 1)
	$A_1$	(1, 0)	(-10, -10)

$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$



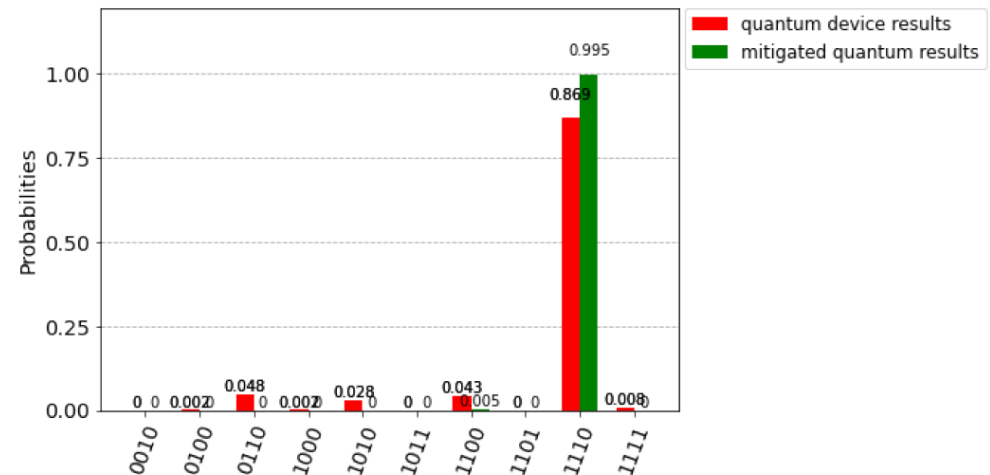
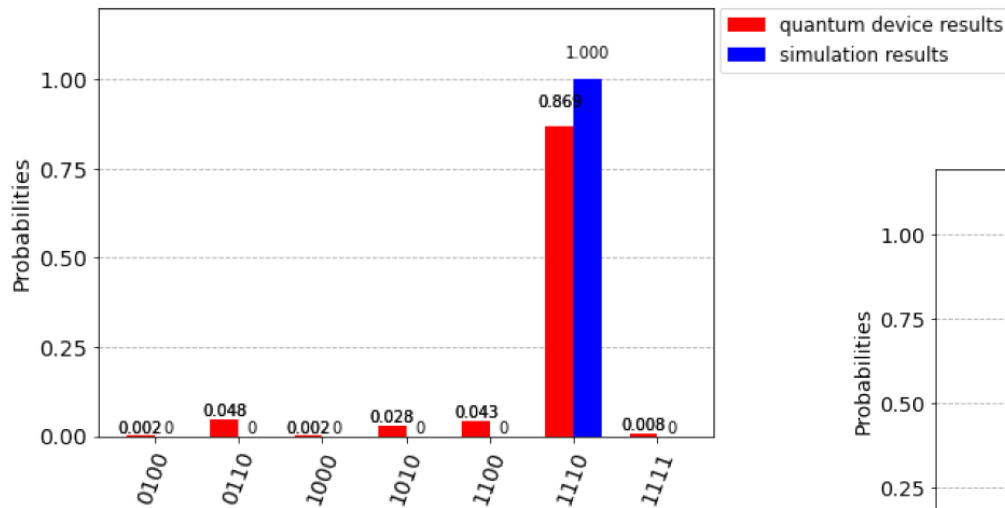
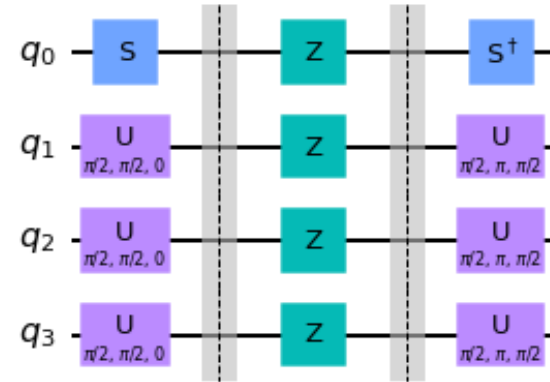
chicken 2		Player B	
Player A		$B_0$	$B_1$
	$A_0$	(4, 4)	(1, 5)
	$A_1$	(5, 1)	(0, 0)

$$\sigma^A = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \sigma^B = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$



# Quantum absentminded driver on IBM-Q

P. Frąckiewicz, K. Rycerz, M. Szopa;  
*Quantum absentminded driver problem revisited*,  
 subm. to Quantum Inf. Processing



# Conclusions

1. Correlated equilibria significantly improve paretoefficiency of Nash equilibria and they can be obtained in quantum games
2. Quantum games give players new strategies not available in classic games and strongly depend on the parameterization used
3. Nash equilibria of quantum in mixed strategies are close to paretoefficiency of correlated equilibria
4. FP parameterization provides a strong isomorphism of the quantum game and gives the same Nash equilibria in mixed strategies as full  $SU(2)$  parameterization of EWL
5. The entanglement of initial state is not necessary to define the quantum absentminded driver model, the key issue is the coherence of quantum evolution